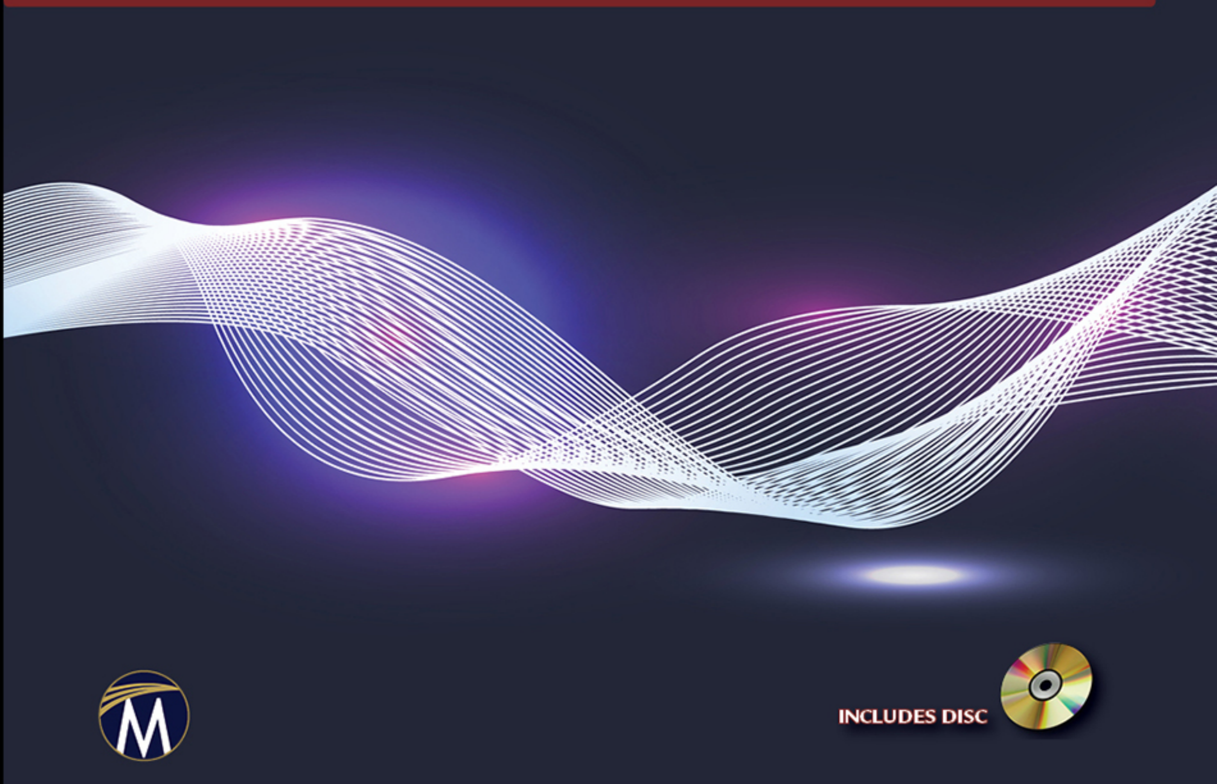
Tensor Analysis for Engineers

Transformations • Applications



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M. TABATABAIAN

Tensor Analysis for Engineers

TENSOR ANALYSIS FOR ENGINEERS

Transformations and Applications

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To my teachers and mentors
for their invaluable transfer of knowledge and direction.

CONTENTS

PREFACE

In engineering and science, physical quantities are often represented by mathematical functions, namely *tensors*. Examples include temperature, pressure, force, mechanical stress, electric/magnetic fields, velocity, enthalpy, entropy, etc. In turn, tensors are categorized based on their rank, i.e. rank zero, one, and so forth. The so-called scalar quantities (e.g. temperature) are tensors of rank zero. Likewise, velocity and force are tensors of rank one, and mechanical stress and gradient of velocity are tensors of rank two. In Euclidean space, which could be of dimension $N = 3, 4, \ldots$, we can define several coordinate systems for our calculation and measurement of physical quantities. For example, in a 3D space, we can define Cartesian, cylindrical, and spherical coordinate systems. In general, we prefer defining a coordinate system whose coordinate surfaces (where one of the coordinate variables is invariant or remains constant) match the physical problem geometry at hand. This enables us to easily define the boundary conditions of the physical problem to the related governing equations, written in terms of the selected coordinate system. This action requires transformation of the tensor quantities and their related derivatives (e.g., gradient, curl, divergence) from Cartesian to the selected coordinate system or vice versa. The topic of *tensor analysis* (also referred to as "tensor calculus," or "Ricci's calculus," since originally developed by Ricci (1835–1925), [1], [2]), is mainly engaged with the definition of tensor-like quantities and their transformation among coordinate systems and others. The topic provides a set of mathematical tools which enables users to perform transformation and calculations of tensors for any well-defined coordinate systems in a systematic way—it is a "machinery." The merit of tensor analysis is to provide a systematic mathematical formulation to derive the general form of the governing equations for arbitrary coordinate systems.

In this book, we aim to provide engineers and applied scientists the tools and techniques of tensor analysis for applications in practical problem solving and analysis activities. The geometry is limited to the Euclidean space/geometry, where the Pythagorean Theorem applies, with well-defined Cartesian coordinate systems as the reference. We discuss quantities defined in curvilinear coordinate systems, like cylindrical, spherical, parabolic, etc. and present several examples and coordinates sketches (some equipped with augmented reality technology) with related calculations. The book is divided into *sections* and *sub-sections* with topics presented in a consistent manner, as given in the table of contents. In addition, we listed several worked-out examples for helping the readers with mastering the topics provided in the prior sections. A list of exercises is provided for further practice for readers.

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July 23, 2018

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Dr. Mehrzad Tabatabaian is a faculty member with several years of teaching experience in the Mechanical Engineering Department, School of Energy at the British Columbia Institute of Technology (BCIT). In addition to teaching courses in the mechanical engineering curriculum, including thermodynamics, energy systems management modeling, and strength of materials, he also does research on renewable energy systems. Dr. Tabatabaian is the School of Energy Research Committee Chair and actively involved in their energy-initiative activities. He has authored several textbooks and published papers in various scientific journals and conferences. He holds several patents in the energy field. Dr. Tabatabaian's recent focus is on wind and solar power which has resulted in registered patents.

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Mehrzad Tabatabaian received his BEng from Sharif University of Technology and advanced degrees from McGill University (MEng and PhD). He has been an active academic, professor, and engineer in leading alternative energy, oil, and gas industries. He has also a Leadership Certificate from the University of Alberta, and holds an APEGBC P.Eng license, and is active member of ASME.

CHAPTER 1

INTRODUCTION

Physical quantities can be represented mathematically by *tensors*. In further sections of this book we will define tensors more rigorously; however, for the introduction we will use this definition. An example of a tensor-like quantity is the temperature in a room (which could be a function of space and time) expressed as a scalar, a tensor of rank zero. Wind velocity is another example, which can be defined when we know both its magnitude or speed—a scalar quantity—and its direction. We define velocity as a vector or a tensor of rank one. Scalars and vectors are familiar quantities to us and we encounter them in our daily life. However, there are quantities, or tensors of rank two, three, or higher that are normally dealt with in technical engineering computations. Examples include mechanical stress in a continuum, like the wall of a pressure vessel—a tensor of rank two—the modulus of elasticity or viscosity in a fluid—tensors of rank four—and so on.

Engineers and scientists calculate and analyze tensor quantities, including their derivatives, using the laws of physics, mostly in the form of governing equations related to the phenomena. These laws must be expressed in an objective form as governing equations, and not subjected to the coordinate system considered. For example, the amount of internal stress in a continuum should not depend on what coordinate system is used for calculations. Sometimes tensors and their involved derivatives in a study must be transformed from one coordinate system to another. Therefore, to satisfy these technical/engineering needs, a mathematical "machinery" is required to perform these operations accurately and systematically between arbitrary coordinate systems. Furthermore, the communication of technical computations requires precise definitions for tensors to guarantee a reliable level of standardization, i.e. identifying true tensor quantities from apparently tensor-like or non-tensor ones. This machinery is called *tensor analysis* [2], [3], [4], [5], [6], [7].

The subject of tensor analysis has two major parts: a) definitions and properties of tensors including their calculus, and b) rules of transformation of tensor quantities among different coordinate systems. For example, consider again the wind velocity vector. By using tensor analysis, we can show that this quantity is a true tensor and transform it from a Cartesian coordinate system to a spherical one, for example. A major outcome of tensor analysis is having general relations for gradient-like operations in arbitrary coordinate systems, including gradient, curl, divergence, Laplacian, etc. which appear in many governing equations in engineering and science. Using tensor notation and definitions, we can write theses governing equations in explicit coordinate-independent forms.

1.1 INDEX NOTATION—THE EINSTEIN SUMMATION CONVENTION

Writing expressions containing tensors could become cumbersome, especially when higher ranked tensors and higher dimensional space are involved. For example, we usually use hatted arrow symbol for vectors (like $\vec{A}$) and bold-font symbols for second rank tensors (like $\boldsymbol{A}$). But this approach is very limited for expressing, for example, the modulus of elasticity, a 4th ranked tensor. Another limitation shows up when writing the components of a tensor in N-dimension space. For example, in 3D we write vector $\vec{A}$ in a Cartesian coordinate system as $\vec{A} = \sum_1^3 A_i\vec{E}_i = A_1\vec{E}_1 + A_2\vec{E}_2 + A_3\vec{E}_3$, where A_i is the component and $\vec{E}_i$ the unit vector. Taking this approach for N-dimension space and carrying the summation symbol (i.e. Σ) is cumbersome and seems unnecessary.

The Einstein summation convention allows us to get rid of the summation symbol, if we carry summation operation for repeated indices in product-type expressions (unless otherwise specified). Using this approach, we can write $\vec{A} = A_i\vec{E}_i$ and a tensor of rank 4, for example, as $\boldsymbol{T} = T_{ijkl}\vec{E}_i\vec{E}_j\vec{E}_k\vec{E}_l$, with all indices range for 1 to N. In practice, however, we even ignore writing the unit vectors and just use the component representing the original tensor; hence this method is also referred to as *component* or *index* notation, or $\vec{A} \equiv A_i$ and $\boldsymbol{T} \equiv T_{ijkl}$. In addition, we represent a tensor's rank by the number of free (i.e. not repeated) indices. We use these definitions and conventions throughout this book.

CHAPTER 2

COORDINATE SYSTEMS DEFINITION

For measuring and calculating physical quantities associated with geometrical points in space we require coordinate systems for reference. These systems (for example, in a 3D space) are composed of surfaces that mutually intersect to specify a geometrical point. For reference, an ideal system of coordinates called a Cartesian system is defined, in Euclidean space, such that it composes of three flat planes. These planes are considered by default to be mutually perpendicular to each other to form an *orthogonal* Cartesian system. In general, the orthogonality is not required to form a coordinate system—this kind of coordinate system is an *oblique* or slanted system. A Cartesian coordinate system, although ideal, is central to engineering and scientific calculations, since it is used as the reference compared to other coordinate systems. It also has properties, as will be shown in further sections, that enable us to calculate the values of tensors [3].

For practical purposes, sometimes it is more convenient to consider curved planes instead of flat ones in a coordinate system. For example, for cases when a cylindrical or a spherical water or oil tank is involved, we prefer that all or some of coordinate surfaces match the geometry of the tank. For a cylindrical coordinate system, we consider the Cartesian system again but replace one of its planes with a cylinder, whereas for a spherical system we replace two flat planes with a sphere and a cone.

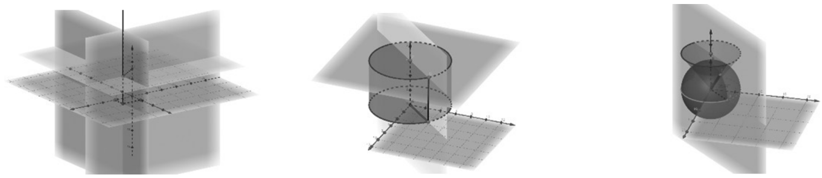


FIGURE 2.1 Sketches of Cartesian, Cylindrical, and Spherical coordinate systems.

Many other coordinate systems are defined/used in practice such as parabolic, bi-spherical, etc. Figure 2.1 shows some examples of common coordinate systems. Animations of the coordinates shown in Figure 2.1 are provided in the accompanying disc.

For organizing the future usage of symbols for arbitrary coordinate systems we define coordinate variables with superscripted letters, for reasons that will be explained in further sections. For Cartesian systems we use y^i and for other arbitrary systems we use x^i—note that i is merely an index, not a raised power. For example, in a 3D space we have $y^1 \equiv X$, $y^2 \equiv Y$, and $y^3 \equiv Z$ where (X, Y, Z) are common notations for Cartesian coordinates. In general, for N dimensional systems we have

$$\begin{cases} y^i \equiv \left(y^1, y^2, y^3, \cdots, y^N\right) & \textit{Cartesian coordinates} \\ x^i \equiv \left(x^1, x^2, x^3, \cdots, x^N\right) & \textit{arbitrary coordinates} \end{cases} \qquad 2.1$$

In this book, we limit the geometry to that known as Euclidean geometry [2], [4], [5]. Therefore, curved space geometry (i.e., Riemann geometry), space-time, and discussions of General Relativity are not included. However, curvilinear coordinate systems and oblique non-orthogonal systems are covered, where applicable, and defined with reference to Euclidean geometry/space.

CHAPTER 3

BASIS VECTORS AND SCALE FACTORS

For the measurement of quantities we need to define metrics or scales in whatever coordinate system we use. These scales are, usually, vectors defined at a point (such as the origin) and are tangent to the coordinate surfaces at that point. These vectors are called *basis vectors.* For Cartesian system y^i we define basis vector $\vec{E}_i$ (see Figure 3.1). Note that subscript i (i.e. $i = 1, 2, \cdots, N$) is merely an index and the reason why we used it as a subscript for basis vectors will be explained in a further section. Now, considering an incremental vector $\overrightarrow{ds}$, from point P to neighboring point P', we define the direction of the basis vector as moving from P to P' or $\vec{E}_i = y^i\big|_P \rightarrow y^i + dy^i\big|_{P'}$. For example, $\vec{E}_1 = y^1\big|_P \rightarrow y^1 + dy^1\big|_{P'}$. Now we define the magnitude of $\vec{E}_i$ such that the magnitude of distance from P to P' is given as

$$ds(i) = \left|dy^i \vec{E}_i\right| \qquad 3.1$$

Also, we can get the distance $\overline{PP'} = ds(i) = dy^i$ in a Cartesian coordinate system. Therefore, the basis vectors magnitude is unity, or $\left|\vec{E}_i\right| = 1$. In other words, the basis vectors in Cartesian system have unit lengths and are well-known unit vectors (usually represented by $\vec{i}, \vec{j}, \vec{k}$). This is more than a trivial result and as shown in further sections, enables us to calculate/measure quantities in other coordinate systems with reference to the Cartesian system. The directed distance vector $\overrightarrow{ds}$ is given as

$$\overrightarrow{ds} = \sum_{i=1}^{N} dy^i \vec{E}_i = dy^i \vec{E}_i \qquad 3.2$$

The magnitude of this vector is the square root of the dot-product of $\overrightarrow{ds}$ with itself,

$$\left|\overrightarrow{ds}\right| = \sqrt{\overrightarrow{ds} \cdot \overrightarrow{ds}} = \sqrt{dy^i dy^j \vec{E}_i \cdot \vec{E}_j} \qquad 3.3$$

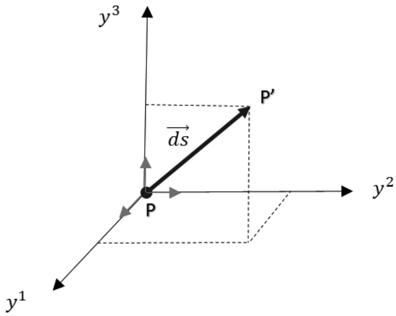


FIGURE 3.1 A Cartesian system with unit vectors and an incremental vector $\overrightarrow{ds}$.

Assuming the Pythagorean theorem holds in Euclidean space we have

$$\left|\overrightarrow{ds}\right| = \sqrt{dy^i dy^i} = \sqrt{\sum_i \left(dy^i\right)^2} \qquad 3.4$$

From Equations 3.3 and 3.4, we can conclude that

$$\vec{E}_i \cdot \vec{E}_j = \begin{cases} 1 \;\; for\, i = j \\ 0 \; for\, i \neq j \end{cases} \qquad 3.5$$

Which also shows that the basis/unit vectors in Cartesian system y^i are mutually perpendicular or the system is orthogonal, as defined.

Now we assume an arbitrary system x^i, which may be neither rectilinear nor orthogonal (see Figure 3.2). In this system the distance between two points—say P and P'—is the same when considering the vector $\overrightarrow{ds}$. In other words, the vector components in system x^i compared to those in system y^i could change along with the basis vector corresponding to system x^i such that the vector itself remains the same, or as an *invariant*. This requirement is simply a statement of independence of physical quantities (and the laws of nature) regardless of the coordinate system we consider for calculation and analysis. Therefore, we have

$$\overrightarrow{ds} = dx^i \vec{e}_i = dy^i \vec{E}_i \qquad 3.6$$

where $\vec{e}_i$ is the basis vector corresponding to system x^i. In general the basis vector $\vec{e}_i$ can vary, both in magnitude and/or direction, from point to point in space. Also, the dx^i may be dimensionless, like the angle coordinate in a polar coordinate system.

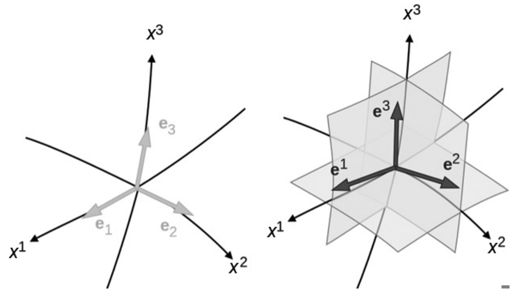


FIGURE 3.2 Sketches of a curvilinear coordinate system and basis vectors[1].

The magnitude of basis vector $\vec{e}_i$ is the scale factor h_i, or

$$h_i = |\vec{e}_i| \tag{3.7}$$

Note that h_i is unity for a Cartesian coordinate system and may be different from unity in general curvilinear systems. We will derive formulae for the calculation of h_i in an arbitrary coordinate system in further sections. The *unit vector* $\vec{e}(i)$ can be defined as

$$\vec{e}(i) = \frac{\vec{e}_i}{h_i} \quad (\text{no summation on } i) \tag{3.8}$$

Obviously, in a Cartesian system the unit vectors and the basis vectors are identical, since h_i is unity.

[1] Common copyright (cc), https://commons.m.wikimedia.org/wiki/File:Vector_1-form.svg#mw-jump-to-license

CHAPTER 4

CONTRAVARIANT COMPONENTS AND TRANSFORMATIONS

For transformations between systems x^i and Cartesian y^i, we must have the functional relationships between their coordinate variables. For example,

$$x^i = F_i\left(y^1, y^2, \cdots, y^N\right) \tag{4.1}$$

gives N number of functions transforming y^i to x^i system. Inversely [4], we can transform from x^i to y^i system using function G_i given as

$$y^i = G_i\left(x^1, x^2, \cdots, x^N\right) \tag{4.2}$$

For example, for a 2D polar coordinate system, $(r,\theta) \equiv \left(x^1, x^2\right)$ with reference to Cartesian system $(X,Y) \equiv \left(y^1, y^2\right)$ we have

$$\begin{cases} X = r\cos\theta \\ Y = r\sin\theta \end{cases} \equiv \begin{cases} y^1 = x^1\cos\left(x^2\right) \\ y^2 = x^1\sin\left(x^2\right) \end{cases}.$$ Or inversely,

$$\begin{cases} r = \sqrt{X^2 + Y^2} \\ \theta = \tan^{-1}\left(\dfrac{Y}{X}\right) \end{cases} \equiv \begin{cases} x^1 = \sqrt{\left(y^1\right)^2 + \left(y^2\right)^2} \\ x^2 = \tan^{-1}\left(\dfrac{y^2}{y^1}\right) \end{cases}.$$

Now by taking partial derivatives of G_i (see Equation 4.2) we can relate the differentials dy^i, in a Cartesian system, to dx^i in an arbitrary coordinate system as

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j \qquad 4.3$$

For example, after expanding Equation 4.3 for $i=1$ we get, $dy^1 = \frac{\partial y^1}{\partial x^1} dx^1 + \frac{\partial y^1}{\partial x^2} dx^2 + \cdots + \frac{\partial y^1}{\partial x^N} dx^N$. The transformation coefficient $\frac{\partial y^i}{\partial x^j}$ can be written in matrix form as $\frac{\partial y^i}{\partial x^j} = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^N} \\ \vdots & \ddots & \vdots \\ \frac{\partial y^N}{\partial x^1} & \cdots & \frac{\partial y^N}{\partial x^N} \end{bmatrix}$. The determinant of the transformation coefficient matrix is defined as the Jacobian of the transformation, or

$$\mathcal{J} = \frac{\partial\left(y^1, y^2, \cdots, y^N\right)}{\partial\left(x^1, x^2, \cdots, x^N\right)} = \begin{vmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^N} \\ \vdots & \ddots & \vdots \\ \frac{\partial y^N}{\partial x^1} & \cdots & \frac{\partial y^N}{\partial x^N} \end{vmatrix} \qquad 4.4$$

The Jacobian can be interpreted as the density of the space. In other words, let's say that we have $\mathcal{J} = 5$ for a given system x^i. This means that we have packed 5 units of Cartesian space into a volume in x^i space through transformation from Cartesian to the given system. The smaller the Jacobian, the smaller the space density would be and vice versa.

Similarly, we can use function F_i (see Equation 4.1) to have

$$dx^i = \frac{\partial x^i}{\partial y^j} dy^j \qquad 4.5$$

It can be shown [4], [2], that the determinant of transformation coefficient $\frac{\partial x^i}{\partial y^j}$ is the inverse of $\mathcal{J}$, or

$$\mathcal{J}^{-1} = \frac{\partial\left(x^1, x^2, \cdots, x^N\right)}{\partial\left(y^1, y^2, \cdots, y^N\right)} = \begin{vmatrix} \dfrac{\partial x^1}{\partial y^1} & \cdots & \dfrac{\partial x^1}{\partial y^N} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x^N}{\partial y^1} & \cdots & \dfrac{\partial x^N}{\partial y^N} \end{vmatrix} \qquad 4.6$$

Equations 4.3 and 4.5 show a pattern for the transformation of differentials dy^i and dx^i, respectively. That is, the partial derivatives of the corresponding coordinates appear in the numerator of the transformation coefficient. Nevertheless, one can ask: does this pattern maintain for general system-to-system transformation? The short answer is "yes" [7]. Hence, for arbitrary systems x^i and x'^i we have

$$dx^i = \frac{\partial x^i}{\partial x'^j} dx'^j \quad \text{and} \quad dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j \qquad 4.7$$

Any quantity, say A^i, that transforms according to Equation 4.7 is defined as a *contravariant* type, with the standard notation of having the index i as a superscript. Therefore, transformation $A^i \rightleftarrows A'^i$ reads

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j \quad \text{and} \quad A^j = \frac{\partial x^j}{\partial x'^i} A'^i \qquad 4.8$$

Obviously, performing the related calculations requires the functional relations between the two systems, i.e., $x^i = func\left(x'^1, x'^2, \cdots, x'^N\right)$ or $x'^i = func\left(x^1, x^2, \cdots, x^N\right)$. This in turn requires having the Cartesian system as a reference for calculating the values of A^j or A'^i, since x^i and x'^i are arbitrary. For example, transforming A^i directly from a spherical to a cylindrical system requires having the functional relations between these coordinates with reference to the Cartesian system as the main reference.

The contravariant component of a vector has a geometrical meaning as well. To show this, we consider a rectilinear non-orthogonal system (x^1, x^2) in 2D, as shown in Figure 4.1. The components of vector $\vec{A}$ can be obtained by drawing parallel lines to the coordinates x^1 and x^2 to find contravariant components A^1 and A^2, respectively.

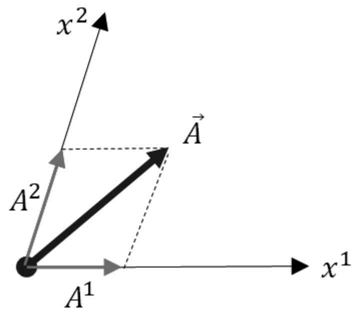


FIGURE 4.1 Contravariant components of a vector in an oblique coordinate system.

With reference to Figure 4.1, we can also find another set of components, A_1 and A_2 of the same vector $\vec{A}$ by drawing perpendicular lines to the coordinates x^1 and x^2. This is shown in Figure 4.2. Obviously, components A_1 and A_2 are different in magnitude from the contravariant components. We define A_1 and A_2 as *covariant* components. In the next section we define the transformation rule for covariant quantities.

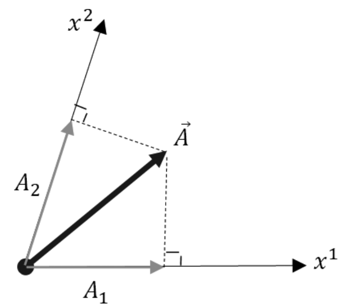


FIGURE 4.2 Covariant components of a vector in an oblique coordinate system.

We can conclude from these definitions that contravariant and covariant components of a vector in a Cartesian system are identical and there is no distinction between them.

CHAPTER 5

COVARIANT COMPONENTS AND TRANSFORMATIONS

We use the standard notation of writing the index i as a subscript for covariant quantities. We consider basis vector $\vec{e}_i$ as a covariant quantity. We can rewrite Equation 3.6, considering two arbitrary systems x^j and x'^k and the fact that $\overrightarrow{ds}$ is coordinate-system independent or invariant, as

$$\overrightarrow{ds} = dx'^k \vec{e}'_k = dx^j \vec{e}_j \tag{5.1}$$

After substituting for dx^j, using Equation 4.7, we get $dx'^k \vec{e}'_k = \frac{\partial x^j}{\partial x'^k} dx'^k \vec{e}_j$ or

$$dx'^k \left(\vec{e}'_k - \frac{\partial x^j}{\partial x'^k} \vec{e}_j \right) = 0 \tag{5.2}$$

Since dx'^k is arbitrary (i.e., the relation is valid for any choice of system x'^k and selections of $dx'^{k=i} \neq 0$ and $dx'^{k \neq i} = 0$, for all values of $i = 1, 2, \cdots, N$) therefore the expression in the bracket must be equal to zero, or we have

$$\vec{e}'_k = \frac{\partial x^j}{\partial x'^k} \vec{e}_j \tag{5.3}$$

All quantities, say A_i, that transform according to Equation 5.3 defined as *covariant* type, with the standard notation of writing the index i as a subscript. Therefore, transformation $A_i \rightleftarrows A'_j$ reads

$$A_i = \frac{\partial x'^j}{\partial x^i} A'_j \text{ and } A'_j = \frac{\partial x^i}{\partial x'^j} A_i \qquad 5.4$$

See Figure 4.2 for the geometrical interpretation of the covariant component of a vector.

CHAPTER 6

PHYSICAL COMPONENTS AND TRANSFORMATIONS

Having defined the contravariant and covariant quantities, like those of the components of a vector, one can ask this question: which one of these two types of components is the actual vector's components in magnitude? The short answer is "none"! However, note that the combination of the contravariant or covariant components and their corresponding basis vectors (contravariant basis vectors will be defined in further section) gives the magnitude of the vector correctly. To find the actual magnitude of the vector, we should find the unit vectors and then the components of the vector corresponding to these unit vectors, or the *physical components*. In other words, the physical components of a vector (or in general, a tensor) are scalars whose physical dimensions and magnitudes are those of the components vectors tangent to the coordinates' lines. Therefore, for vector $\vec{A}$, if we designate $A(i)$ as its physical component and the corresponding unit vector as $\vec{e}(i)$ we can write, then

$$\vec{A} = A^i\vec{e}_i = A(i)\vec{e}(i) \tag{6.1}$$

Now, using Equation 3.8 and substituting for $\vec{e}_i$ we can write $A^i\vec{e}(i)h_i = A(i)\vec{e}(i)$. Therefore,

$$A(i) = A^i h_i, \text{ (no summation on } i) \tag{6.2}$$

Or $A(1) = A^1 h_1$, $A(2) = A^2 h_2$, and so forth. The scale factor h_i is the parameter that turns the contravariant component into the physical component, when multiplied by it. Again, we can conclude that in a Cartesian system the physical and contravariant/covariant components are identical, since $h_i = 1$.

Up to this point we have defined contravariant, covariant, and physical components of a vector along with covariant and physical/unit basis vectors. In principle, we can represent a vector by any combination of its components with the corresponding basis vectors. This requires having the contravariant basis vector definition. We will define this quantity and how it is transformed in a further section. But first, we must have the definition of a tensor, as well as a specific tensor called a *metric tensor*.

CHAPTER 7

TENSORS—MIXED AND METRIC

In previous sections, we defined vectors or tensors of rank one. Higher order tensors could be defined based on similar definitions. For example, a tensor of rank two requires more than one free index, like mechanical stress or strain tensors. In general, we have the contravariant components written as A^{ij} for tensor A. We avoid placing double-arrows over the symbol for tensors, for simplicity and generality. Since, in principle, we can have tensors of rank N, designating them with N number of hatted arrows is not a practical exercise. The invariant quantity is written as

$$A = A^{ij}\vec{e}_i\vec{e}_j \qquad 7.1$$

Since A is invariant, its value remains the same regardless of the coordinate system used. We transform A from an arbitrary coordinate system x'^i to another system, say x^i, or $A'^{ij}\vec{e}'_i\vec{e}'_j = A^{km}\vec{e}_k\vec{e}_m$. Now, using Equation 5.3, we transform the covariant basis vectors $\vec{e}_k$ and $\vec{e}_m$ to system x'^i. Therefore, we have $A^{km}\vec{e}_k\vec{e}_m = A^{km}\frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^m}\vec{e}'_i\vec{e}'_j$. Using these last two relations, we get $\vec{e}'_i\vec{e}'_j\left(A'^{ij} - \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^m}A^{km}\right) = 0$. Therefore, since basis vectors are non-zero, we have

$$A'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^m}A^{km} \qquad 7.2$$

Quantities that transform according to Equation 7.2 are defined as doubly contravariant tensors, of rank two. Similarly, we can define mixed tensors, for example of rank two, as

$$A'^{i}_{j} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{m}}{\partial x'^{j}} A^{k}_{m} \tag{7.3}$$

The rule for transformation of tensors is like those given for vectors, i.e., the contravariant component of the tensor transforms like a contravariant vector and the covariant component like a covariant vector. Also, we can expand the definition to a doubly covariant tensor, like

$$A'_{ij} = \frac{\partial x^{k}}{\partial x'^{i}} \frac{\partial x^{m}}{\partial x'^{j}} A_{km} \tag{7.4}$$

Equations 7.2–7.4 are useful relations defining second rank tensors and their transformations. Expanding this rule, we can write the transformation of a N-rank tensor, for example contravariant components, as

$$A'^{ij\cdots N} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x'^{j}}{\partial x^{m}} \cdots \frac{\partial x'^{N}}{\partial x^{M}} A^{km\cdots M} \tag{7.5}$$

In an arbitrary curvilinear system, the Pythagorean theorem applies but does not appear in the same form as it does in flat Cartesian systems. This is mainly due to the fact that basis vectors in an arbitrary system change in magnitude and/or direction, from point to point. Therefore, measuring the distance ds between two points on a curved surface, like a sphere, for example, requires scaling the coordinates by multiplying them with some scalar coefficients. These coefficients are the components of a tensor, called a *metric tensor,* associated with the coordinate system selected. Recalling Equation 3.6, we can find the square of the magnitude of differential distance $\overrightarrow{ds}$ by dot-product operation, or $ds^2 = \overrightarrow{ds} \cdot \overrightarrow{ds} = \left(dx^i \vec{e}_i\right) \cdot \left(dx^j \vec{e}_j\right)$. The result is

$$ds^2 = dx^i dx^j \left(\vec{e}_i \cdot \vec{e}_j\right) \tag{7.6}$$

The term in the bracket, on the R.H.S of Equation 7.6, is a scalar (i.e., a tensor of rank zero) resulting from a dot-product operation on the basis vectors. Hence, we can write it as

$$\vec{e}_i \cdot \vec{e}_j = \left|\vec{e}_i\right| \left|\vec{e}_j\right| \cos\alpha = h_i h_j \cos\alpha \tag{7.7}$$

where α is the angle between the tangents to x^i and x^j axes of the coordinate system at the selected point in space. Geometrically speaking, Equation 7.7 is the multiplication of the magnitude of a basis vector with the projection of the other one along the direction of the original one. Therefore, this quantity could be used to identify whether a system (or at least the basis vectors selected) is orthogonal or not. Also, it could be a metric for measuring the distance over a curved surface. It has more properties and applications, which we will encounter in future sections; hence, it is designated by a symbol and a name, i.e., *metric tensor* g_{ij}

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j \tag{7.8}$$

It is easily seen from Equation 7.8 that a metric tensor is symmetric since order in a dot-product is irrelevant (i.e., commutative). Also, if $g_{ij} = 0$ for $i \neq j$ then the coordinates of the system x^i are orthogonal. For example, the metric tensor in a 3D orthogonal system can be represented by a 3×3 matrix containing null off-diagonal elements,

$$g_{ij} = \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix}, \textit{orthogonal system} \tag{7.9}$$

It would be useful to apply metric tensor definitions and expand Equation 7.6 using Equations 7.7 and 7.9 for an orthogonal 3D system, which gives

$$ds^2 = h_1^2\left(dx^1\right)^2 + h_2^2\left(dx^2\right)^2 + h_3^2\left(dx^3\right)^2 \tag{7.10}$$

For example, in a Cartesian system we recover the familiar form of the Pythagorean theorem (i.e., $ds^2 = (dX)^2 + (dY)^2 + (dZ)^2$), since $g_{11} = g_{22} = g_{33} = \underbrace{\vec{E}_i \cdot \vec{E}_i}_{\textit{no sum on } i} = 1$. Similarly, for a spherical coordinate system (r, φ, θ) we have $h_1^2 = g_{11} = 1$, $h_2^2 = g_{22} = r^2$, and $h_3^2 = g_{33} = r^2 \sin^2 \varphi$ (see Example 8.2 for scale factors). Therefore, we get $ds^2 = (dr)^2 + r^2 (d\varphi)^2 + r^2 \sin^2 \varphi \, (d\theta)^2$.

So far, we have not shown that g_{ij} is a tensor. To do this, we transform the metric tensor to another arbitrary coordinate system, say x'^i. We then can write $g'_{ij} = \vec{e}'_i \cdot \vec{e}'_j = \dfrac{\partial x^k}{\partial x'^i}\dfrac{\partial x^m}{\partial x'^j}\underbrace{\left(\vec{e}_k \cdot \vec{e}_m\right)}_{=g_{km}}$. Therefore, we have $g'_{ij} = \dfrac{\partial x^k}{\partial x'^i}\dfrac{\partial x^m}{\partial x'^j} g_{km}$; that is, when compared to Equation 7.4 we can conclude that g_{km} is a doubly covariant tensor.

As mentioned in the previous sections, to calculate or measure tensor-like quantities we use a Cartesian system as the reference. To evaluate g_{ij} we use Equation 7.8 and transform the covariant basis vector $\vec{e}_i$ to the Cartesian system, y^i. Therefore, $g_{ij} = \vec{e}_i \cdot \vec{e}_j = \frac{\partial y^k}{\partial x^i}\frac{\partial y^m}{\partial x^j}\left(\vec{E}_k \cdot \vec{E}_m\right)$. But the quantity $\vec{E}_k \cdot \vec{E}_m$, or the dot-product of the basis/unit vectors in the Cartesian system, is either one or zero, due to orthogonality of the coordinates. Hence $\vec{E}_k \cdot \vec{E}_m = \begin{cases} 1, \text{ when } k = m \\ 0, \text{ when } k \neq m \end{cases}$. Using this property, we can write

$$g_{ij} = \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j} \tag{7.11}$$

Using Equation 7.11, we can readily conclude that the metric tensor in a Cartesian system is the familiar Kronecker delta, $\delta_{ij} = \vec{E}_i \cdot \vec{E}_j$. For 3D Cartesian coordinates, we get a diagonal unity matrix $\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

It can be shown ([3], [4]) that the determinant of a metric tensor is equal to the square of the Jacobian of the transformation matrix, or

$$g = \left|g_{ij}\right| = \mathcal{J}^2 \tag{7.12}$$

Therefore, in an orthogonal system x_i, having $g_{ij} = 0$ for $i \neq j$, we get

$$g = g_{11}g_{22}\cdots g_{NN} = h_1^2 h_1^2 \cdots h_N^2, \text{orthogonal system} \tag{7.13}$$

Note that $g_{ii} = \left|\vec{e}_i\right|\left|\vec{e}_i\right| = h_i^2$ (no sum on i) (see Equation 7.7). From Equations 7.12 and 7.13 we can conclude that the Jacobian for an orthogonal system is equal to the products of the scale factors, or

$$\mathcal{J} = h_1 h_2 \cdots h_N, \text{orthogonal system} \tag{7.14}$$

For example, in a spherical coordinate system the Jacobian is $\mathcal{J}_{spherical} = h_r h_\varphi h_\theta = r^2 \sin\varphi$ (see Example 8.2 for scale factors).

CHAPTER 8

METRIC TENSOR OPERATION ON TENSOR INDICES

An important and useful application of metric tensor is that it can be used to lower and raise indices of a tensor. Therefore, we can change a contravariant component of a tensor to a covariant one, or vice versa, by multiplying the appropriate metric tensor to the tensor at hand. For example, let's define the covariant component of a vector (i.e., a tensor of rank one) as $A_i = A^j g_{ij}$, where A^j is the given contravariant component. Now, we show that the quantity $A^j g_{ij}$ is transformed like a covariant quantity, according to Equation 5.4. Considering systems x^i and x'^i, we can write $A'^j g'_{ij} = \left(\frac{\partial x'^j}{\partial x^k} A^k\right)\left(\frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j} g_{mn}\right) = \left(\frac{\partial x'^j}{\partial x^k}\frac{\partial x^n}{\partial x'^j}\right)\frac{\partial x^m}{\partial x'^i} A^k g_{mn}$. Note that the expression $\frac{\partial x'^j}{\partial x^k}\frac{\partial x^n}{\partial x'^j} = \frac{\partial x^n}{\partial x^k} = \delta^n_k$, where $\delta^n_k = \begin{cases} 1, n = k \\ 0, n \neq k \end{cases}$ is a mixed second rank tensor (a Kronecker delta). Therefore, we have $\left(\frac{\partial x'^j}{\partial x^k}\frac{\partial x^n}{\partial x'^j}\right)\frac{\partial x^m}{\partial x'^i} A^k g_{mn} = \frac{\partial x^m}{\partial x'^i}\underbrace{\left(\delta^n_k A^k\right)}_{=A^n} g_{mn} = \frac{\partial x^m}{\partial x'^i}\left(A^n g_{mn}\right)$. But the expression in the last bracket is actually A_m, according to our definition. Finally, we have $A'_i = A'^j g'_{ij} = \frac{\partial x^m}{\partial x'^i} A_m$, which clearly shows that the quantity $A'^j g'_{ij}$ is transformed like a covariant component (see Equation 5.4). Therefore, we have, also using the symmetry property of a metric tensor or $g_{ij} = g_{ji}$,

$$A_i = A^j g_{ij} = A^j g_{ji} \qquad 8.1$$

Using Equation 8.1, we can write the expression for a contravariant basis vector, or

$$\vec{e}_i = \vec{e}^{\,j} g_{ij} \tag{8.2}$$

which shows that the metric tensor lowers the contravariant index, while the dummy index is dropped out. This will enable us to write the vector $\vec{A}$ in terms of covariant or contravariant basis vectors, as

$$\vec{A} = A^i \vec{e}_i = A^i \left(g_{ij} \vec{e}^{\,j} \right) = \left(A^i g_{ij} \right) \vec{e}^{\,j} = A_j \vec{e}^{\,j} \tag{8.3}$$

Note that by using physical components and corresponding unit vectors, we can write

$$\vec{A} = A^i \vec{e}_i = A_j \vec{e}^{\,j} = A(k)\vec{e}(k) \tag{8.4}$$

So far, we have defined the doubly covariant metric tensor g_{ij}. Similarly, the doubly contravariant metric tensor is defined $g^{ij} = \vec{e}^{\,i} \cdot \vec{e}^{\,j}$. Therefore, we can write

$$\vec{A} = A^i \vec{e}_i = A_j g^{ij} \vec{e}_i = A_j \vec{e}^{\,j}$$

Also, the combination of the two types of metric tensor gives the mixed one, as

$$g^{ij} g_{jk} = \delta^i_k \tag{8.5}$$

Equation 8.5 can be derived as follow: we have $A^i = g^{ij} A_j = g^{ij} \underbrace{g_{jk} A^k}_{=A_j}$. In this expression, we write $A^i = \delta^i_k A^k$, therefor we have $A^k \left(\delta^i_k - g^{ij} g_{jk} \right) = 0$, but since A^k is arbitrary the expression in the bracket is zero. Hence $g^{ij} g_{jk} = \delta^i_k$.

With reference to Figure 3.2, Figure 4.1, Figure 4.2, and Equation 8.3 we can sketch the vector $\vec{A}$ in terms of its covariant/contravariant components with their corresponding basis vectors, as shown in Figure 8.1 (see Example 15.3). Users may also want to watch a related video at

https://www.youtube.com/watch?v=CliW7kSxxWU.

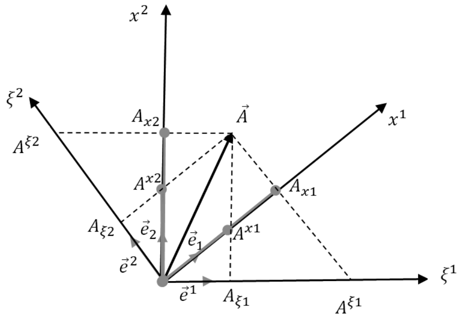


FIGURE 8.1 Sketch for covariant and contravariant components of a vector and basis vectors in a non-orthogonal coordinate system x^i and alternate system ξ^i.

In the following subsections, we present two examples demonstrating related calculations for cylindrical and spherical coordinate systems.

8.1 EXAMPLE: CYLINDRICAL COORDINATE SYSTEMS

Consider a cylindrical polar coordinate system $\left(x^1,x^2,x^3\right) \equiv (r,\theta,z)$ where r is the radial distance, θ the azimuth angle, and z the elevation from $X-Y$ plane, w.r.t. the Cartesian coordinates $\left(y^1,y^2,y^3\right) \equiv (X,Y,Z)$, as shown in Figure 8.2. The functional relations corresponding to cylindrical and Cartesian systems are:

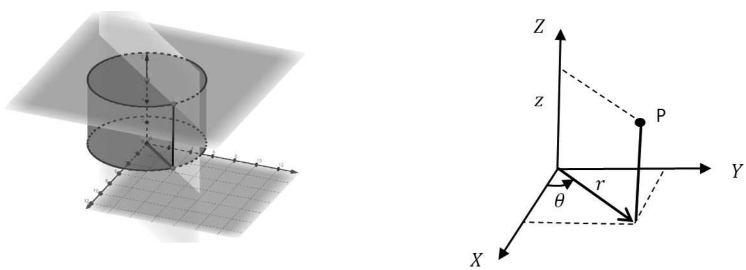


FIGURE 8.2 Cylindrical coordinate system.

$\begin{cases} X = r\cos\theta \\ Y = r\sin\theta \\ Z = z \end{cases} \equiv \begin{cases} y^1 = x^1 \sin x^2 \\ y^2 = x^1 \sin x^2 \\ y^3 = x^3 \end{cases}$. Find the inverse relations (i.e., $x^i = F_i\left(y^1, y^2,\cdots,y^N\right)$, the basis vectors $\vec{e}_i$ and $\vec{e}^{\,i}$ for the cylindrical coordinate system in terms of the Cartesian unit vectors. Also find the scale factors, unit vectors, and metric tensors g_{ij} and g^{ij} and line element magnitude $\left|\overrightarrow{ds}\right|$.

Solution:

The functions F_i can be obtained by solving for X, Y, Z, using the functional relation, given

$$\begin{cases} r = \sqrt{X^2 + Y^2} \\ \theta = \tan^{-1}\left(\dfrac{Y}{X}\right) \\ z = Z \end{cases} \equiv \begin{cases} x^1 = \sqrt{\left(y^1\right)^2 + \left(y^2\right)^2} \\ x^2 = \tan^{-1}\left(\dfrac{y^1}{y^2}\right) \\ x^3 = Y^3 \end{cases}$$

. To find the covariant basis vectors, we use $\vec{e}_i = \dfrac{\partial y^j}{\partial x^i}\vec{E}_j$. Therefore, we get $\vec{e}_1 = \dfrac{\partial y^1}{\partial x^1}\vec{E}_1 + \dfrac{\partial y^2}{\partial x^1}\vec{E}_2 + \dfrac{\partial y^3}{\partial x^1}\vec{E}_3 = \cos\theta\,\vec{E}_1 + \sin\theta\,\vec{E}_2$. Similarly, $\vec{e}_2 = \dfrac{\partial y^1}{\partial x^2}\vec{E}_1 + \dfrac{\partial y^2}{\partial x^2}\vec{E}_2 + \dfrac{\partial y^3}{\partial x^2}\vec{E}_3 = -r\sin\theta\,\vec{E}_1 + r\cos\theta\,\vec{E}_2$, and $\vec{e}_3 = \dfrac{\partial y^1}{\partial x^3}\vec{E}_1 + \dfrac{\partial y^2}{\partial x^3}\vec{E}_2 + \dfrac{\partial y^3}{\partial x^3}\vec{E}_3 = \vec{E}_3$. In terms of coordinates notation, we have

$$\begin{cases} \vec{e}_r = \cos\theta\,\vec{E}_1 + \sin\theta\,\vec{E}_2 \\ \vec{e}_\theta = -r\sin\theta\,\vec{E}_1 + r\cos\theta\,\vec{E}_2 \\ \vec{e}_z = \vec{E}_3 \end{cases}$$

For calculating contravariant basis vectors, we use $\vec{e}^{\,i} = \dfrac{\partial x^i}{\partial y^j}\vec{E}_j$. After expansion and using the functional relations, we get

$$\begin{cases} \vec{e}^{\,r} = \cos\theta\,\vec{E}_1 + \sin\theta\,\vec{E}_2 \\ \vec{e}^{\,\theta} = -\dfrac{\sin\theta}{r}\vec{E}_1 + \dfrac{\cos\theta}{r}\vec{E}_2 \\ \vec{e}^{\,z} = \vec{E}_3 \end{cases}$$

The scale factors are the magnitudes of the covariant basis vectors. Hence $h_1 = h_r = \sqrt{\vec{e}_r \cdot \vec{e}_r} = 1$, $h_2 = h_\theta = \sqrt{\vec{e}_\theta \cdot \vec{e}_\theta} = r$, and $h_3 = h_z = 1$. The unit vectors $\vec{e}(i) = \dfrac{\vec{e}_i}{h_i}$, are

$$\begin{cases} \vec{e}(r) = \cos\theta\,\vec{E}_1 + \sin\theta\,\vec{E}_2 \\ \vec{e}(\theta) = -\sin\theta\,\vec{E}_1 + \cos\theta\,\vec{E}_2 \\ \vec{e}(z) = \vec{E}_3 \end{cases}$$

Note that unit vectors $\vec{e}(r)$ and $\vec{e}(\theta)$ in cylindrical coordinate system change with location and are not constant vectors.

The metric tensor is $g_{ij} = \vec{e}_i \cdot \vec{e}_j$, or $g_{rr} = h_r^2 = 1$, $g_{\theta\theta} = h_\theta^2 = r^2$, $g_{zz} = h_z^2 = 1$, and the off-diagonal elements of the metric tensor are null, hence the coordinate system is orthogonal, or $g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The Jacobian is $\mathcal{J} = h_r h_\theta h_z = r$.

The contravariant metric tensor can be calculated using $g^{ij} = \vec{e}^i \cdot \vec{e}^j$, or $g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is equal to the inverse of covariant metric tensor.

Another way for calculating the contravariant basis vectors is by using the relation $\vec{e}^i = g^{ij}\vec{e}_j$, or $\vec{e}^1 = g^{11}\vec{e}_1 = \vec{e}_1$, $\vec{e}^2 = g^{22}\vec{e}_2 = r^{-2}\vec{e}_2$, and $\vec{e}^3 = g^{33}\vec{e}_3 = \vec{e}_3$.

For the line element we have, $\left|\overrightarrow{ds}\right| = \sqrt{\left(h_i dx^i\right)^2} = \sqrt{(dr)^2 + (rd\theta)^2 + (dz)^2}$.

8.2 EXAMPLE: SPHERICAL COORDINATE SYSTEMS

Consider a spherical polar coordinate system $\left(x^1, x^2, x^3\right) \equiv (r, \varphi, \theta)$ where r is the radial distance, φ the polar/meridian angle, and θ the azimuthal angle w.r.t. the Cartesian coordinates $\left(y^1, y^2, y^3\right) \equiv (X, Y, Z)$, as shown in Figure 8.3. The functional relations corresponding to spherical and Cartesian systems are:

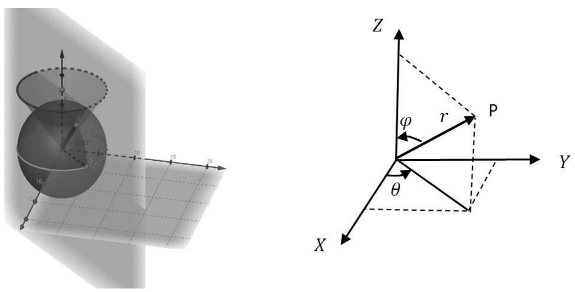


FIGURE 8.3 Spherical coordinate system.

$\begin{cases} X = r\sin\varphi\cos\theta \\ Y = r\sin\varphi\sin\theta \\ Z = r\cos\varphi \end{cases} \equiv \begin{cases} y^1 = x^1\sin x^2\cos x^3 \\ y^2 = x^1\sin x^2\sin x^3 \\ y^3 = x^1\cos x^2 \end{cases}$. Find the inverse relations (i.e., $x^i = F_i\left(y^1, y^2, \cdots, y^N\right)$, the basis vectors $\vec{e}_i$ and $\vec{e}^{\,i}$ for the spherical coordinate system in terms of the Cartesian unit vectors. Also find the scale factors and unit vectors and metric tensors g_{ij} and g^{ij} and line element magnitude $\left|\overrightarrow{ds}\right|$.

Solution:

The functions F_i can be obtained by solving for X, Y, Z, using corresponding functional relations, or $\begin{cases} r = \sqrt{X^2+Y^2+Z^2} \\ \varphi = \cos^{-1}\left(\dfrac{Z}{\sqrt{X^2+Y^2+Z^2}}\right) \\ \theta = \tan^{-1}\left(\dfrac{Y}{X}\right) \end{cases} \equiv \begin{cases} x^1 = \sqrt{\left(y^1\right)^2+\left(y^2\right)^2+\left(y^3\right)^2} \\ x^2 = \cos^{-1}\left(\dfrac{y^3}{\sqrt{\left(y^1\right)^2+\left(y^2\right)^2+\left(y^3\right)^2}}\right) \\ x^3 = \tan^{-1}\left(\dfrac{y^1}{y^2}\right) \end{cases}$.

To find the covariant basis vectors, we use $\vec{e}_i = \dfrac{\partial y^j}{\partial x^i}\vec{E}_j$. Therefore, we get $\vec{e}_1 = \dfrac{\partial y^1}{\partial x^1}\vec{E}_1 + \dfrac{\partial y^2}{\partial x^1}\vec{E}_2 + \dfrac{\partial y^3}{\partial x^1}\vec{E}_3 = \sin x^2\cos x^3\vec{E}_1 + \sin x^3\sin x^2\vec{E}_2 + \cos x^2\vec{E}_3$ Similarly, $\vec{e}_2 = \dfrac{\partial y^1}{\partial x^2}\vec{E}_1 + \dfrac{\partial y^2}{\partial x^2}\vec{E}_2 + \dfrac{\partial y^3}{\partial x^2}\vec{E}_3 = x^1\cos x^2\cos x^3\vec{E}_1 + x^1\sin x^3\cos x^2\vec{E}_2 - x^1\sin x^2\vec{E}_3$, and $\vec{e}_3 = \dfrac{\partial y^1}{\partial x^3}\vec{E}_1 + \dfrac{\partial y^2}{\partial x^3}\vec{E}_2 + \dfrac{\partial y^3}{\partial x^3}\vec{E}_3 = -x^1\sin x^2\sin x^3\vec{E}_1 + x^1\cos x^3\sin x^2\vec{E}_2$. In terms of coordinate variables, we have

$$\begin{cases} \vec{e}_r = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3 \\ \vec{e}_\varphi = r\cos\varphi\cos\theta\vec{E}_1 + r\cos\varphi\sin\theta\vec{E}_2 - r\sin\varphi\vec{E}_3 \\ \vec{e}_\theta = -r\sin\varphi\sin\theta\vec{E}_1 + r\sin\varphi\cos\theta\vec{E}_2 \end{cases}$$

The scale factors are the magnitudes of the basis vectors. Hence $h_1 = h_r = \sqrt{\vec{e}_r \cdot \vec{e}_r} = 1$, $h_2 = h_\varphi = \sqrt{\vec{e}_\varphi \cdot \vec{e}_\varphi} = r$, and $h_3 = h_\theta = \sqrt{\vec{e}_\theta \cdot \vec{e}_\theta} = r\sin\varphi$.
The unit vectors $\vec{e}(i) = \dfrac{\vec{e}_i}{h_i}$, are

$$\begin{cases} \vec{e}(r) = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3 \\ \vec{e}(\varphi) = \cos\varphi\cos\theta\vec{E}_1 + \cos\varphi\sin\theta\vec{E}_2 - \sin\varphi\vec{E}_3 \\ \vec{e}(\theta) = -\sin\theta\vec{E}_1 + \cos\theta\vec{E}_2 \end{cases}$$

Note that unit vectors in spherical coordinate system change with location and are not constant vectors. The metric tensor is $g_{ij} = \vec{e}_i \cdot \vec{e}_j$, or $g_{rr} = h_r^2 = 1$, $g_{\varphi\varphi} = h_\varphi^2 = r^2$, $g_{\theta\theta} = h_\theta^2 = r^2\sin^2\varphi$, and the off-diagonal elements of the metric tensor are null, hence the coordinate system is orthogonal, $g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & (r\sin\varphi)^2 \end{bmatrix}$. The Jacobian is $\mathcal{J} = h_r h_\varphi h_\theta = r^2\sin\varphi$. The $g^{ii} = h_i^{-2}$, or $g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & (r\sin\varphi)^{-2} \end{bmatrix}$, then we can calculate the contravariant basis vectors, as $\vec{e}^j = g^{ij}\vec{e}_i$, or $\vec{e}^1 = \vec{e}_1$, $\vec{e}^2 = \dfrac{1}{r^2}\vec{e}_2$, and $\vec{e}^3 = \dfrac{1}{r^2\sin^2\varphi}\vec{e}_3$ which yields

$$\begin{cases} \vec{e}^r = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3 \\ \vec{e}^\varphi = \dfrac{\cos\varphi\cos\theta}{r}\vec{E}_1 + \dfrac{\cos\varphi\sin\theta}{r}\vec{E}_2 - \dfrac{\sin\varphi}{r}\vec{E}_3 \\ \vec{e}^\theta = -\dfrac{\sin\theta}{r\sin\varphi}\vec{E}_1 + \dfrac{\cos\theta}{r\sin\varphi}\vec{E}_2 \end{cases}$$

For the line element we have, $\left|\overline{ds}\right| = \sqrt{\left(h_i dx^i\right)^2} = \sqrt{(dr)^2 + (rd\varphi)^2 + (r\sin\varphi d\theta)^2}$.

CHAPTER 9

DOT AND CROSS PRODUCTS OF TENSORS

We often must multiply tensors by each other. We consider two vectors for discussion here, without losing generality. When we multiply two scalars (i.e., tensors of rank zero) we just deal with arithmetic multiplication. But a vector has a direction, in addition to its magnitude, that should be taken into consideration for multiplication operations. In principle we can have two vectors just multiply by each other like $\vec{A}\vec{B}$, which is a tensor of rank two, called a *dyadic* product. Or, multiply the vectors' magnitudes and form a new vector with the resulting magnitude directed perpendicular to the plane containing the original vectors, the *cross-product* $\vec{A}\times\vec{B}$, which is a tensor of rank one. Or, multiply the vectors' magnitudes and reduce the rank of the product by projecting one vector on the other one, the *dot-product* $\vec{A}\cdot\vec{B}$, which is a tensor of rank zero. Obviously, for higher than rank-one tensors we would have a greater number of combinations as the result of their products. In general, compared to the rank of original tensors, dyadic product increases the rank, cross-product keeps the rank, and dot-product decreases the rank, i.e., the result is a tensor quantity of higher, the same, or lower rank (by one level/rank) w.r.t to the original quantities, respectively.

We start with dot-product operations. Let's consider two vectors $\vec{A}$ and $\vec{B}$; the dot-product of these two vectors is given by $\vec{A}\cdot\vec{B}=\left(A^i\vec{e}_i\right)\cdot\left(B^j\vec{e}_j\right)$, using the contravariant components of the vectors. We expand this expression, with Equations 7.7 and 7.8, to get

$$\vec{A}\cdot\vec{B}=A^iB^jg_{ij}=A^iB^jh_ih_j\cos\alpha \qquad 9.1$$

where α is the angle between the two vectors. Further, using Equation 8.1 we get

$$\vec{A} \cdot \vec{B} = A^i B_i = A_j B^j \qquad 9.2$$

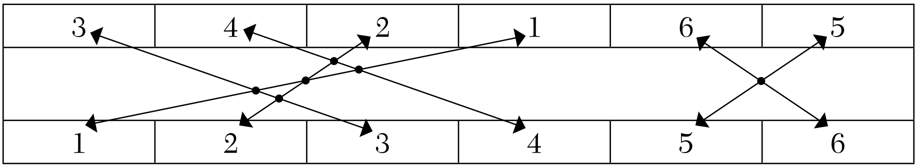
Similarly, we could start with using the covariant components of the vector, but the conclusive results would be the same as the expression given by Equation 9.2. Readers may want to do this as an exercise. From Equation 9.2 we can conclude that the order of multiplication does not change the result of the dot-product, or $A^i B_i = B_i A^i$.

In an orthogonal Cartesian system, either Equation 9.1 or 9.2 would result in $\vec{A} \cdot \vec{B} = A^i B^j \delta_{ij} = A^1 B^1 + A^2 B^2 + \cdots + A^N B^N = A_1 B_1 + A_2 B_2 + \cdots + A_N B_N$. The second equality results from the fact that in orthogonal Cartesian systems there is no distinction between contravariant and covariant components. An example of the dot-product quantity of two vectors is the mechanical work—the dot product of force and distance vectors.

The cross-product operation can be established with a more general formulation using the permutation symbol (also referred to as the Levi-Civita symbol) defined as

$$e_{i_1 i_2 \cdots i_N} = \begin{cases} +1 & \textit{even permutation} \\ -1 & \textit{odd permutation} \\ 0 & \textit{repeated index} \end{cases} \qquad 9.3$$

For example, for $N = 6$ we get $e_{i_1 i_2 \cdots i_6}$. The permutation is considered even if an even number of interchanges of indices put the indices in arithmetic order (i.e., 123456), and is considered odd if an odd number of interchanges of indices put the indices in order. For any case with a repeated index the permutation symbol is zero. As an example, let's consider e_{342165}. Examining the indices, we find out that 4 interchanges (i.e. $342165 \xrightarrow{(3\leftrightarrow 1)} 142365 \xrightarrow{(4\leftrightarrow 2)} 124365 \xrightarrow{(4\leftrightarrow 3)} 123465 \xrightarrow{(6\leftrightarrow 5)} 123456$) put them in order; note that the sequence of interchanges is irrelevant. Therefore $e_{342165} = 1$. Similarly, $e_{362145} = -1$, since 5 interchanges put the indices in order. If any of the indices is repeated then it is zero, for example $e_{344165} = e_{142165} = 0$, etc. We purposefully called the permutation symbol a *symbol* and not a tensor, for reasons that will be explained later in this section. Another way of identifying the even or odd number of permutations is to write down the given indices as a set and connect them with the equivalent number set but in arithmetic order. Then the number of points at the cross-section of the lines connecting the same numbers is the order of permutation. This method is shown in the following figure for e_{342165}. The connecting lines intersect at six points; hence the permutation is even.



Now considering two vectors $\vec{A}$ and $\vec{B}$ in a Cartesian system the cross-product of these two vectors is another vector $\vec{C}$, given as (we propose this expression, for now at least)

$$C^i = \underbrace{e_{ijk}A^jB^k}_{A\times B} = -\underbrace{e_{ijk}B^jA^k}_{B\times A}, \textit{Cartesian} \qquad 9.4$$

for the i-component of $\vec{C}$. This relation clearly shows that the order of multiplication matters for cross-product operation. For example, in a 3D Cartesian system we have $\vec{A}=\left(A^1,A^2,A^3\right)$ and $\vec{B}=\left(B^1,B^2,B^3\right)$ and we find their cross-product using Equation 9.4, $\vec{C}=\left(C^1,C^2,C^3\right)=\vec{A}\times\vec{B}$ as, $C^1 = A^2B^3 - A^3B^2$, $C^2 = A^3B^1 - A^1B^3$, and $C^3 = A^1B^2 - A^2B^1$.

Note that in Equation 9.4, we have written the relation for the contravariant component of the resulting vector $\vec{C}$. As we mentioned previously, in a Cartesian system contravariant or covariant components need not be considered; hence we usually place all indices as subscripts. But in an arbitrary system we should make sure that the resulting C^i is transformed like a contravariant component/quantity. To show this, we set guidelines to form a tensor expression in an arbitrary coordinate system, x^i as follows:

- **a.** Identify the rank of the expression representing the desired quantity in terms of a tensor; i.e., zero is a scalar, one is a vector, two or more a tensor of the corresponding rank.
- **b.** Identify and decide if the expression should be in contravariant or covariant form and place the indices appropriately (i.e., as subscript for covariant and superscript for contravariant).
- **c.** Form the expression based on guidelines (a) and (b).

d. Transform the expression from arbitrary system x^i to another system, say x'^i (or vice versa) using the linear operators $\dfrac{\partial x'^i}{\partial x^j}$ or $\dfrac{\partial x^i}{\partial x'^j}$ according to the combination rule (see Sections 4 and 5).

e. Show that the resulting expression recovers the related familiar form in Cartesian coordinates.

Usually, starting from an expression form in a Cartesian system (as listed in part (e), above) gives a reasonable and usually the correct form to start with. Let's apply these guidelines to the expression given by Equation 9.4. By observing the expression, we conclude that the contravariant components A^j and B^k are correct forms since they transform like contravariant quantities. However, for the permutation symbol we need to make sure that it is a tensor quantity, i.e., it transforms like a tensor. It can be shown that the permutation symbol does not transform like a tensor unless a necessary adjustment is implemented [2], [4], [7] by multiplying it by the Jacobian of the transformation (see Equation 4.4). Now, we define the covariant and contravariant *permutation tensors* by multiplying Jacobian and inverse of Jacobian to the permutation symbol, respectively, given as

$$\begin{cases} \mathcal{E}_{i_1 i_2 \cdots i_N} = \mathcal{J} e_{i_1 i_2 \cdots i_N} \\ \mathcal{E}^{i_1 i_2 \cdots i_N} = \mathcal{J}^{-1} e_{i_1 i_2 \cdots i_N} \end{cases} \tag{9.5}$$

Therefore, the general expression for the vector $\vec{C}$, for example, is $C_i = \mathcal{J} e_{ijk} A^j B^k$ and $C^i = \mathcal{J}^{-1} e_{ijk} A_j B_k$. Using permutation tensor, we will get

$$\begin{cases} C_i = \mathcal{E}_{ijk} A^j B^k = \mathcal{J} e_{ijk} A^j B^k \quad \textit{covariant component} \\ C^i = \mathcal{E}^{ijk} A_j B_k = \mathcal{J}^{-1} e_{ijk} A_j B_k \; \textit{contravariant component} \end{cases} \tag{9.6}$$

Note that in Equation 9.6, for the covariant (contravariant) component of the resulting vector $\vec{C}$, we used the combination of covariant (contravariant) permutation tensor and contravariant (covariant) components of the two contributing vectors $\vec{A}$ and $\vec{B}$.

To show that, for example, $\mathcal{E}_{ijk} A^j B^k$ transforms like a covariant quantity, we write down its transformation between two arbitrary systems, or $\mathcal{E}'_{ijk} A'^j B'^k = \left(\mathcal{E}_{lmn} \dfrac{\partial x^l}{\partial x'^i} \dfrac{\partial x^m}{\partial x'^j} \dfrac{\partial x^n}{\partial x'^k} \right) \left(A^p \dfrac{\partial x'^j}{\partial x^p} \right) \left(B^q \dfrac{\partial x'^k}{\partial x^q} \right) = \mathcal{E}_{lmn} A^p B^q \dfrac{\partial x^l}{\partial x'^i} \delta^m_p \delta^n_q =$ $\left(\mathcal{E}_{lmn} A^m B^n \right) \dfrac{\partial x^l}{\partial x'^i}$. Similarly, for the contravariant form (i.e., the second expression in Equation 9.6) it can be shown that it does indeed transform like a contravariant quantity. Readers may want to do this as an exercise.

The cross-product of two vectors can be written in terms of their physical components as well. This can be done using, for example, the contravariant component when multiplied by the corresponding scale factor. Therefore, we can write for an orthogonal system (using Equations 9.5, 9.6 and 11.8)

$$\underbrace{h_i C^i}_{C(i)} = h_i \mathcal{E}^{ijk} A_j B_k \text{, or } C(i) = \frac{h_i e_{ijk}}{\mathcal{J}} A_j B_k = \frac{h_i e_{ijk}}{\mathcal{J}} \left[h_j h_k A(j) B(k) \right] \text{, or}$$

$$C(i) = \frac{e_{ijk}}{\mathcal{J}} h_i h_j h_k \left[A(j) B(k) \right] \text{,} \textit{orthogonal} \qquad 9.7$$

For orthogonal coordinate systems, using $\mathcal{J} = h_1 h_2 h_3$, we get

$$\begin{cases} (\vec{A} \times \vec{B})(1) = C(1) = [A(2)B(3) - A(3)B(2)] \\ (\vec{A} \times \vec{B})(2) = C(2) = [A(3)B(1) - A(1)B(3)] \\ (\vec{A} \times \vec{B})(3) = C(3) = [A(1)B(2) - A(2)B(1)] \end{cases} \qquad 9.8$$

Expanding on these results, in a N-dimensional space, the generalized cross-product of $N-1$ vectors reads, using Equation 9.6,

$$\begin{cases} C_{i_1} = \mathcal{E}_{i_1 i_2 \cdots i_N} A^{i_2} B^{i_3} \cdots Z^{i_N} \\ C^{i_1} = \mathcal{E}^{i_1 i_2 \cdots i_N} A^{i_2} B^{i_3} \cdots Z^{i_N} \end{cases} \qquad 9.9$$

where now vector $\vec{C}$ is perpendicular to the hyperplane formed by the $N-1$ vectors $\vec{A}$ through $\vec{Z}$.

Immediately, using Equation 9.9 we can conclude that in an arbitrary system the unit volume $d\mathbb{V}$ is, [2],

$$d\mathbb{V} = \mathcal{J} dx^1 dx^2 \cdots dx^N \qquad 9.10$$

For example, in the spherical system (r, φ, θ) we have $d\mathbb{V} = r^2 \sin\varphi \, dr d\varphi \, d\theta$. Another useful extension is for the orthogonal systems (i.e., $g_{i \neq j} = 0$), for which we get $|g_{ij}| = \prod g_{ii} = g_{11} g_{22} \cdots g_{NN}$. But $g_{11} = \vec{e}_1 \cdot \vec{e}_1 = h_1^2$, $g_{22} = \vec{e}_2 \cdot \vec{e}_2 = h_2^2$ and so forth. Hence, $|g_{ij}| = h_1^2 h_2^2 \cdots h_N^2$. Using the relationship $|g_{ij}| = \mathcal{J}^2$, we finally reach at

$$\mathcal{J} = \prod_i h_i = h_1 h_2 \cdots h_N \ \textit{orthogonal system} \qquad 9.11$$

And hence

$$d\mathbb{V} = \prod_i h_i dx^i = (h_1 dx^1)(h_2 dx^2) \cdots (h_N dx^N) \ \textit{orthogonal system} \qquad 9.12$$

9.1 DETERMINANT OF AN $N \times N$ MATRIX USING PERMUTATION SYMBOLS

Using the permutation symbol e_{ijk} (see equation 9.3), we can write the determinant of a 3×3 matrix $[M]$, as $|M| = e_{ijk} M_{1i} M_{2j} M_{3k} = e_{ijk} M_{i1} M_{j2} M_{k3}$, given by expansion based on row or column, respectively, and M_{ij} represent the elements of the matrix [2], [4]. To expand on these results, we consider $[M]$ to be an $N \times N$ matrix, $(N \geq 3)$. Therefore, we can write its determinant as

$$|M| = e_{i_1 i_2 \cdots i_N} M_{1i_1} M_{2i_2} \cdots M_{Ni_N} = e_{i_1 i_2 \cdots i_N} M_{i_1 1} M_{i_2 2} \cdots M_{i_N N} \qquad 9.13$$

See the application example in Section 15.10.

CHAPTER 10

GRADIENT VECTOR OPERATOR—CHRISTOFFEL SYMBOLS

In engineering and science, most if not all governing equations (e.g., equilibrium, Navier-Stokes, Maxwell's) contain terms that involve derivatives of tensor quantities. These equations are mathematical models of related quantity transports, like momentum, mass, energy, etc. Different forms of derivatives result from the gradient operator, which is a vector that takes the derivative of and performs dyadic, dot-product, cross-product, etc. operations on the involved tensors. Depending on the rank of the tensors under the gradient operation, the result could be complex expressions. In the previous sections, we have defined and derived necessary formulae to tackle the derivation of gradient vector transformation and consequently find the general forms of related derivatives appearing in the governing equations.

10.1 COVARIANT DERIVATIVES OF VECTORS—CHRISTOFFEL SYMBOLS OF THE 2ND KIND

The covariant component of the gradient vector $\vec{\nabla}$ is defined as $\nabla_i = \dfrac{\partial}{\partial x^i}$, or we can write the full vector as $\vec{\nabla} = \vec{e}^{\,i} \dfrac{\partial}{\partial x^i} = \vec{e}^{\,i} \nabla_i$. To show that ∇_i transforms like a covariant component between two arbitrary systems, we can write $\nabla'_i = \dfrac{\partial}{\partial x'^i} = \dfrac{\partial x^j}{\partial x'^i}\dfrac{\partial}{\partial x^j} = \dfrac{\partial x^j}{\partial x'^i}\nabla_j$. This expression clearly shows that ∇'_i is a covariant quantity. The transformation from x'^i to x^i is obvious.

When a gradient operates on a scalar, or tensor, of rank zero like temperature, pressure, or concentration, we can write it as $\nabla_i \psi$ and the corresponding transformation is $\nabla'_i \psi' = \nabla'_i \psi = \frac{\partial x^j}{\partial x'^i} \nabla_j \psi$, using the fact that scalar ψ is invariant (or $\psi = \psi'$) and independent of the coordinate systems selected. Now, assuming that the gradient operates on a vector, say $\vec{A}$, which is also an invariant quantity, the result is $\vec{\nabla}\vec{A}$, or a tensor of rank two. The covariant component is then $\nabla_i \vec{A}$, which is a vector by itself. Expanding, we get

$$\nabla_i \vec{A} = \nabla_i \left(A^j \vec{e}_j \right) = \vec{e}_j \left(\nabla_i A^j \right) + A^j \left(\nabla_i \vec{e}_j \right) \qquad 10.1$$

The first term in the R.H.S of Equation 10.1 is the straight-forward derivative of the component A^j. But the second term involves a gradient of the basis vector $\vec{e}_j$, which is not necessarily zero in an arbitrary system since the basis vector's magnitude and/or direction may change from point to point. Note that in a Cartesian system, $\nabla_i \vec{E}_j = 0$, i.e., the second term in Equation 10.1, vanishes. Therefore, we can interpret $\nabla_i \vec{e}_j$ as a measure of the curvature of the arbitrary coordinate surface $x^i = constant$ (e.g., in a spherical coordinate system $X^2 + Y^2 + Z^2 = r^2$ is the surface of the sphere). The physical meaning of $\nabla_i \vec{e}_j$ is this that it represents the rate of change of $\vec{e}_j$ with respect to coordinate variable x^i in the direction of basis vector $\vec{e}_k$. This quantity is represented by Γ^k_{ij} (some authors use $\begin{Bmatrix} k \\ ij \end{Bmatrix}$ instead) and is the *Christoffel symbol of the second kind* [8].

It may be useful to pause at this point and make sure that the physical/geometrical meaning of the Christoffel symbol is well understood. To help with this understanding, we consider a simple 2D polar coordinate system, $(r, \theta) \equiv (x^1, x^2)$ as shown in Figure 10.1.

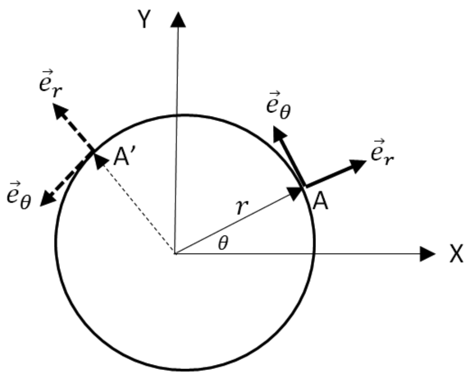


FIGURE 10.1 2D Polar coordinate system.

As shown, the covariant basis vectors $\vec{e}_r$ and $\vec{e}_\theta$ change from point to point in space, e.g., points A and A', in general. Let's consider, for example $\nabla_\theta \vec{e}_\theta = \frac{\partial \vec{e}_\theta}{\partial \theta}$ which represents the change for basis vector $\vec{e}_\theta$ with respect to coordinate variable θ. This derivative is a vector and has two components in the (r,θ) coordinate system. Therefore, we can write it as a linear combination of the basis vectors in polar coordinate system $\vec{e}_r$ and $\vec{e}_\theta$, or

$$\nabla_\theta \vec{e}_\theta = \frac{\partial \vec{e}_\theta}{\partial \theta} = \alpha\, \vec{e}_r + \beta\, \vec{e}_\theta \qquad 10.2$$

Similar expressions can be written down for other possible derivatives (i.e., $\nabla_\theta \vec{e}_r$, $\nabla_r \vec{e}_\theta$, $\nabla_r \vec{e}_r$). In our example (see Equation 10.2) the coefficients α and β are the corresponding Christoffel symbols, also referred to as *connection coefficients*. α is the component of the vector $\frac{\partial \vec{e}_\theta}{\partial \theta}$ in the direction of $\vec{e}_r$, or $\Gamma^r_{\theta\theta}$ and β is the component of the vector $\frac{\partial \vec{e}_\theta}{\partial \theta}$ in the direction of $\vec{e}_\theta$, or $\Gamma^\theta_{\theta\theta}$. Using these definitions, we can write all possible derivatives with their corresponding Christoffel symbols as

$$\begin{cases} \dfrac{\partial \vec{e}_\theta}{\partial \theta} = \Gamma^r_{\theta\theta} \vec{e}_r + \Gamma^\theta_{\theta\theta} \vec{e}_\theta \\ \dfrac{\partial \vec{e}_r}{\partial \theta} = \Gamma^r_{r\theta} \vec{e}_r + \Gamma^\theta_{r\theta} \vec{e}_\theta \\ \dfrac{\partial \vec{e}_\theta}{\partial r} = \Gamma^r_{\theta r} \vec{e}_r + \Gamma^\theta_{\theta r} \vec{e}_\theta \\ \dfrac{\partial \vec{e}_r}{\partial r} = \Gamma^r_{rr} \vec{e}_r + \Gamma^\theta_{rr} \vec{e}_\theta \end{cases} \qquad 10.3$$

The values of Christoffel symbols can be calculated using the functional relations between the polar and Cartesian systems. For this purpose, we need the relations for the basis vectors. Using Equation 5.3, we have $\vec{e}_r = \frac{\partial X}{\partial r}\vec{E}_1 + \frac{\partial Y}{\partial r}\vec{E}_2$ and $\vec{e}_\theta = \frac{\partial X}{\partial \theta}\vec{E}_1 + \frac{\partial Y}{\partial \theta}\vec{E}_2$, where $\vec{E}_i$ are unit vectors in Cartesian system. Performing the calculations, using the functional relation between the polar and Cartesian systems, as $X = r\cos\theta$ and $Y = r\sin\theta$, we get

$$\begin{cases} \vec{e}_r = \cos\theta\, \vec{E}_1 + \sin\theta\, \vec{E}_2 \\ \vec{e}_\theta = -r\sin\theta\, \vec{E}_1 + r\cos\theta\, \vec{E}_2 \end{cases} \qquad 10.4$$

Using Equations 10.4 we get $\frac{\partial \vec{e}_\theta}{\partial \theta} = -r\cos\theta \vec{E}_1 - r\sin\theta \vec{E}_2 = -r\vec{e}_r$, $\frac{\partial \vec{e}_r}{\partial \theta} = -\sin\theta \vec{E}_1 + \cos\theta \vec{E}_2 = \frac{1}{r}\vec{e}_\theta$, $\frac{\partial \vec{e}_\theta}{\partial r} = -\sin\theta \vec{E}_1 + \cos\theta \vec{E}_2 = \frac{1}{r}\vec{e}_\theta$, and $\frac{\partial \vec{e}_r}{\partial r} = 0$. Finally, comparing these results with Equation 10.3, we find the Christoffel symbols values for polar coordinate system as $\Gamma^r_{\theta\theta} = -r$, $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$, and $\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{rr} = \Gamma^r_{\theta r} = \Gamma^r_{r\theta} = \Gamma^r_{rr} = 0$. Table 10.1 gives the summary of these results and the matrix form of these symbols for polar coordinate systems.

TABLE 10.1 Christoffel symbols of the 2nd kind for a 2D polar coordinate system (r,θ).

Symbol	Meaning	Value	Symmetry	Matrix form
$\Gamma^r_{\theta\theta}$	$\vec{e}_\theta$ change w.r.t θ in $\vec{e}_r$ direction	$-r$	–	$\Gamma^r = \begin{bmatrix} \Gamma^r_{rr} & \Gamma^r_{r\theta} \\ \Gamma^r_{\theta r} & \Gamma^r_{\theta\theta} \end{bmatrix}$
$\Gamma^r_{\theta r}$	$\vec{e}_\theta$ change w.r.t r in $\vec{e}_r$ direction	0	$=\Gamma^r_{r\theta}$	
Γ^r_{rr}	$\vec{e}_r$ change w.r.t r in $\vec{e}_r$ direction	0	–	
$\Gamma^\theta_{\theta\theta}$	$\vec{e}_\theta$ change w.r.t θ in $\vec{e}_\theta$ direction	0	–	$\Gamma^\theta = \begin{bmatrix} \Gamma^\theta_{rr} & \Gamma^\theta_{r\theta} \\ \Gamma^\theta_{\theta r} & \Gamma^\theta_{\theta\theta} \end{bmatrix}$
$\Gamma^\theta_{r\theta}$	$\vec{e}_r$ change w.r.t θ in $\vec{e}_\theta$ direction	$1/r$	$=\Gamma^\theta_{\theta r}$	
Γ^θ_{rr}	$\vec{e}_r$ change w.r.t r in $\vec{e}_\theta$ direction	0	–	

The number of independent Christoffel symbols can be obtained using $\frac{N^2(N+1)}{2}$, for an N-dimensional system. For example, in a $2D$ polar coordinate system (r,θ) with $N=2$, we get $\frac{4\times 3}{2} = 6$ symbols, among them only two are non-zero. Note that Christoffel symbols are symmetric with respect to the lower indices (see the next section).

Now we continue with our general formulation and discussion, after establishing the physical/geometrical meaning of Christoffel symbols of 2nd kind with this example.

Rewriting $\nabla_i \vec{e}_j$ using the definition of Christoffel symbol (see Equation 10.3), we have

$$\nabla_i \vec{e}_j = \Gamma^k_{ij} \vec{e}_k \qquad 10.5$$

Plugging back into Equation 10.1, we get $\nabla_i\vec{A} = \vec{e}_j\left(\nabla_i A^j\right) + A^j\Gamma_{ij}^k\vec{e}_k$. In this relation, for the last term we interchange j and k indices, since they are dummy indices. Then, $\nabla_i\vec{A} = \vec{e}_j\left(\nabla_i A^j\right) + A^k\Gamma_{ik}^j\vec{e}_j = \vec{e}_j\left(\nabla_i A^j + \Gamma_{ik}^j A^k\right)$. The expression in the bracket is a tensor, since both $\nabla_i\vec{A}$ and $\vec{e}_j$ are tensors. We define the expression in the bracket as $A_{,i}^j$ where the comma in the subscript indicates differentiation and index i represents the covariant component of gradient vector. Hence,

$$A_{,i}^j = \frac{\partial A^j}{\partial x^i} + \Gamma_{ik}^j A^k \tag{10.6}$$

Equation 10.6 is the covariant derivative of the contravariant component of vector $\vec{A}$. Or

$$\nabla_i\vec{A} = A_{,i}^j\vec{e}_j \tag{10.7}$$

10.2 CONTRAVARIANT DERIVATIVES OF VECTORS

At this point in our discussion, it seems logical to seek the contravariant component of gradient vector, ∇^i. We can write this operator as $\nabla^i = g^{ij}\nabla_j$ using a metric tensor (see Section 8). Therefore, we can raise the derivative index by multiplying the contravariant metric tensor to the expression, or

$$A_{,\cdot}^{k\,i} = g^{ij}A_{,j}^k \tag{10.8}$$

Equation 10.8 is the contravariant derivative of the contravariant component of vector $\vec{A}$. Or

$$\nabla^i\vec{A} = A_{,\cdot}^{k\,i}\vec{e}_k \tag{10.9}$$

From the start we could use the covariant component of vector $\vec{A}$, to find the corresponding covariant and contravariant gradient components as well (see Equation 10.1). To this end, we can write the covariant derivative of the covariant component of vector $\vec{A}$, as

$$\nabla_i\vec{A} = \nabla_i\left(A_j\vec{e}^j\right) = \vec{e}^j\left(\nabla_i A_j\right) + A_j\left(\nabla_i\vec{e}^j\right) \tag{10.10}$$

The first term on the R.H.S of Equation 10.10 is straight-forward, the derivative of the component A_j. But the second term involves the gradient of the contravariant basis vector $\vec{e}^{\,j}$, for which we require a new expression. To find this quantity, we use Equation 10.6 to write $\vec{e}^{\,j}_{,i} = \dfrac{\partial \vec{e}^{\,j}}{\partial x^i} + \Gamma^j_{ik}\vec{e}^{\,k}$. But due the fact that $\vec{e}^{\,j}_{,i}$ is a tensor (this can be shown by transformation of the terms involved between arbitrary systems, like $A^j_{,i}$) and its value in a Cartesian system is null, hence it is equal to zero in any arbitrary coordinate system. Therefore, $\vec{e}^{\,j}_{,i} = \dfrac{\partial \vec{e}^{\,j}}{\partial x^i} + \Gamma^j_{ik}\vec{e}^{\,k} = 0$, or

$$\nabla_i \vec{e}^{\,j} = \frac{\partial \vec{e}^{\,j}}{\partial x^i} = -\Gamma^j_{ik}\vec{e}^{\,k} \qquad 10.11$$

Substituting for $\nabla_i \vec{e}^{\,j}$ from Equation 10.11, into Equation 10.10, we get $\nabla_i\left(A_j\vec{e}^{\,j}\right) = \vec{e}^{\,j}\left(\nabla_i A_j\right) - A_j\left(\Gamma^j_{ik}\vec{e}^{\,k}\right)$. In this relation for the last term we interchange j and k indices, since they are dummy indices. Then, after factoring out $\vec{e}^{\,j}$, we get $\nabla_i\left(A_j\vec{e}^{\,j}\right) = \vec{e}^{\,j}\left(\nabla_i A_j - A_k\Gamma^k_{ij}\right)$. We define the expression in the bracket as $A_{j,i}$ where the comma in subscript indicates differentiation and i represents the covariant component of gradient vector. Hence

$$A_{j,i} = \frac{\partial A_j}{\partial x^i} - \Gamma^k_{ij}A_k \qquad 10.12$$

Equation 10.12 is the covariant derivative of the covariant component of vector $\vec{A}$. Or

$$\nabla_i \vec{A} = A_{j,i}\vec{e}^{\,j} \qquad 10.13$$

10.3 COVARIANT DERIVATIVES OF A MIXED TENSOR

Comparing Equations 10.6 and 10.12, we observe that for each index a term involving a Christoffel symbol is added to the expression on the R.H.S and when a covariant component of a vector is involved in a derivative operation, a minus sign is multiplied to the corresponding Christoffel symbol, contrary to a plus sign for when the contravariant component is involved. This rule can be used to extend the derivative calculation of a tensor of higher ranks. To show this, we consider a mixed tensor of third rank, for example A^{ij}_k with the invariant $A = A^{ij}_k\vec{e}_i\vec{e}_j\vec{e}^{\,k}$. The covariant derivative can be written as

$$A^{ij}_{k,n} = \frac{\partial A^{ij}_k}{\partial x^n} + \Gamma^i_{nm}A^{jm}_k + \Gamma^j_{nm}A^{im}_k - \Gamma^m_{nk}A^{ij}_m \qquad 10.14$$

Therefore, $\nabla_n A = A^{ij}_{k,n}\vec{e}_i\vec{e}_j\vec{e}^{\,k}$ which is the component n of a tensor of rank four (i.e., $\vec{\nabla}A = A^{ij}_{k,n}\vec{e}_i\vec{e}_j\vec{e}^{\,k}\vec{e}^{\,n}$).

10.4 CHRISTOFFEL SYMBOL RELATIONS AND PROPERTIES—1ST AND 2ND KINDS

The definition of a Christoffel symbol of the first kind, properties, and some relations for calculation of Christoffel symbols are discussed in this section. The properties are:

1. Symmetry of the lower indices: The Christoffel symbol of the second kind is symmetric w.r.t lower indices, or

$$\Gamma_{ij}^{k} = \Gamma_{ji}^{k} \tag{10.15}$$

To show this, we use Equation 10.5, after changing index $k \to n$, and perform a dot product operation on it by contravariant basis vector $\vec{e}^{k}$. Or $\vec{e}^{k} \cdot \nabla_i \vec{e}_j = \Gamma_{ij}^{n} \underbrace{\vec{e}^{k} \cdot \vec{e}_n}_{\delta_n^k} = \Gamma_{ij}^{k}$. Therefore, we can write

$$\Gamma_{ij}^{k} = \vec{e}^{k} \cdot \left(\nabla_i \vec{e}_j\right) \tag{10.16}$$

We can use Equation 10.16 to calculate Christoffel symbol Γ_{ij}^{k} in Cartesian coordinate system variables y^i. After expanding the R.H.S expression by appropriate transformation and using chain rule, we will get $\vec{e}^{k} \cdot \left(\nabla_i \vec{e}_j\right) = \left(\frac{\partial x^k}{\partial y^m}\vec{E}^m\right) \cdot \frac{\partial y^n}{\partial x^i}\frac{\partial}{\partial y^n}\left(\frac{\partial y^p}{\partial x^j}\vec{E}_p\right) = \frac{\partial x^k}{\partial y^m}\frac{\partial y^n}{\partial x^i}\frac{\partial}{\partial y^n}\frac{\partial y^p}{\partial x^j}\underbrace{(\vec{E}^m \cdot \vec{E}_p)}_{\delta_p^m} =$ $\frac{\partial x^k}{\partial y^p}\underbrace{\left(\frac{\partial y^n}{\partial x^i}\frac{\partial x^q}{\partial y^n}\right)}_{\delta_i^q}\frac{\partial^2 y^p}{\partial x^q \partial x^j}$. Where, in the last expression we inserted ($\frac{\partial x^q}{\partial x^q} = 1$, or using chain rule). Finally, we get

$$\Gamma_{ij}^{k} = \frac{\partial x^k}{\partial y^p}\frac{\partial^2 y^p}{\partial x^i \partial x^j} \tag{10.17}$$

Equation 10.17 clearly shows the symmetry of Γ_{ij}^{k} for its lower indices, since the order of differentiation is irrelevant, i.e. $\frac{\partial^2 y^p}{\partial x^i \partial x^j} = \frac{\partial^2 y^p}{\partial x^j \partial x^i}$.

2. Christoffel symbol is not a tensor: Because its value in a Cartesian system is zero; hence, if it transforms as a tensor it should be zero in any arbitrary system as well. But we know that this is not the case (see Equations 10.10 and 10.17).

3. Covariant derivative of the metric tensor: This quantity leads to a useful formula for calculating the value of a Christoffel symbol. First, we find the covariant derivative of the metric tensor, or $\nabla_i g_{jk} = \nabla_i \left(\vec{e}_j \cdot \vec{e}_k\right) = \vec{e}_j \cdot (\nabla_i \vec{e}_k) + \left(\nabla_i \vec{e}_j\right) \cdot \vec{e}_k$. Now, using Equation 10.5, we can write $\nabla_i \vec{e}_k = \Gamma^n_{ik} \vec{e}_n$ and $\nabla_i \vec{e}_j = \Gamma^n_{ij} \vec{e}_n$. Therefore, $\nabla_i g_{jk} = \Gamma^n_{ik} \underbrace{(\vec{e}_j \cdot \vec{e}_n)}_{g_{jn}} + \Gamma^n_{ij} \underbrace{(\vec{e}_n \cdot \vec{e}_k)}_{g_{nk}}$ and finally we get

$$\nabla_i g_{jk} = \frac{\partial g_{jk}}{\partial x^i} = \Gamma^n_{ik}\, g_{jn} + \Gamma^n_{ij}\, g_{nk} \qquad 10.18$$

Note that by comparing Equation 10.18 with the rule of covariant derivatives of tensors (see Equation 10.14) we can conclude that $g_{jk,i} = 0$, a reasonable result considering that a metric tensor in a Cartesian system is a constant/unity and hence its derivative is equal to zero in an arbitrary system.

4. Christoffel symbol of the first kind: So far, we have defined the Christoffel symbol of the second kind. We now derive formulas that can be used for calculating its value as well as defining a Christoffel symbol of the first kind. To do this, we manipulate Equation 10.18 by permuting the indices (i.e., $i \to j, j \to k, \text{and}\, k \to i$), twice in sequence. Hence we get two alternative but equivalent relations as $\frac{\partial g_{ki}}{\partial x^j} = \Gamma^n_{ji}\, g_{jn} + \Gamma^n_{jk}\, g_{ni}$ and $\frac{\partial g_{ij}}{\partial x^k} = \Gamma^n_{kj}\, g_{in} + \Gamma^n_{ki}\, g_{nj}$. Now we subtract Equation 10.18 from the sum of the latter two relations, using along the way the symmetry of both metric tensor and Christoffel symbols. The result reads

$$\Gamma^n_{ij}\, g_{nk} = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right] \qquad 10.19$$

In Equation 10.19, the quantity $\Gamma^n_{ij}\, g_{nk}$ is the *Christoffel symbol of the first kind*, also written as $\Gamma_{ij,k} \equiv \left[ij,k\right]$. We can use $\Gamma^n_{ij}\, g_{nk}$ to find a relation for calculation of the Christoffel symbol of the second kind in terms of metric tensor. To do this we multiply it by g^{mk} to get $\Gamma^n_{ij}\, g_{nk} g^{mk} = \delta_n^{\ m} \Gamma^n_{ij} = \Gamma^m_{ij}$. Therefore, after multiplying Equation 10.19 by g^{mk}, we get

$$\Gamma^m_{ij} = \frac{g^{mk}}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right] \qquad 10.20$$

Equation 10.20 states that the Christoffel symbol of the second kind is equal to the first kind multiplied by the doubly contravariant metric tensor related to the coordinate system in use. Observing Equation 10.19, a symmetry shows up. That is, the derivative is w.r.t. the coordinates indices and for each term in the bracket the remaining indices are used for the metric tensor.

5. For an orthogonal coordinate system we can simplify Equation 10.20, using the property of g_{ij} and g^{ij} being diagonal tensors or matrices (see Equation 7.9). In other words, we can write $g_{ij} = g^{ij} = 0$ for $i \neq j$. Therefore, examining the terms on the R.H.S of Equation 10.20, we can conclude that $\frac{\partial g_{ij}}{\partial x^k} = 0$ and $\frac{\partial g_{ik}}{\partial x^j} \neq 0$ only when $i = k$. Also $g^{mk} \neq 0$ when $m = k$ and $\frac{\partial g_{jk}}{\partial x^i} \neq 0$, since j cannot be equal to k. By implementing these relations among the indices (i.e., $i = k = m$ and $i \neq j$) and rewriting Equation 10.20, we get

$$\Gamma^i_{ij} = \frac{g^{ii}}{2}\frac{\partial g_{ii}}{\partial x^j}, \qquad \textit{no summmation on i} \qquad 10.21$$

Equation 10.8, for $i = j \neq k$ reduces to

$$\Gamma^k_{ii} = -\frac{g^{kk}}{2}\frac{\partial g_{ii}}{\partial x^k}, \qquad \textit{no summmation on i and k} \qquad 10.22$$

For orthogonal coordinate systems, using Equation 7.13, we substitute for $g_{ii} = h_i^2$ and $g^{ii} = h_i^{-2}$ in Equation 10.21 to get $\Gamma^i_{ij} = \frac{h_i^{-2}}{2}\frac{\partial\left(h_i^2\right)}{\partial x^j}$, or

$$\Gamma^i_{ij} = \frac{1}{h_i}\frac{\partial h_i}{\partial x^j}, \qquad \textit{no summmation on i}, (\textit{orthogonal}) \qquad 10.23$$

Simply, by letting $i = j$ in Equation 10.23, we get

$$\Gamma^i_{ii} = \frac{1}{h_i}\frac{\partial h_i}{\partial x^i}, \qquad \textit{no summmation on i}, (\textit{orthogonal}) \qquad 10.24$$

Similarly, Equation 10.22 yields $\Gamma^k_{ii} = -\frac{h_k^{-2}}{2}\frac{\partial\left(h_i^2\right)}{\partial x^k}$, or

$$\Gamma^k_{ii} = -\frac{h_i}{h_k^2}\frac{\partial h_i}{\partial x^k}, \qquad \textit{no summmation on i and k}, (\textit{orthogonal}) \qquad 10.25$$

6. Another useful relation can be derived for the value of Γ^i_{in}, which is $\vec{e}_i$ change w.r.t x^n in the direction of $\vec{e}_i$. We use the fact that the derivative of the permutation tensor, or $\mathcal{E}^{ijk}_{,n}$ is zero, since its value in the Cartesian coordinates is null. Using Equation 10.14, we can write $\mathcal{E}^{ijk}_{,n} = \dfrac{\partial \mathcal{E}^{ijk}}{\partial x^n} + \Gamma^i_{nm}\mathcal{E}^{mjk} + \Gamma^j_{nm}\mathcal{E}^{imk} + \Gamma^k_{nm}\mathcal{E}^{ijm} = 0$. To simplify the derivation, without losing generality, we select $i=1$, $j=2$, $k=3$. Hence, we get $\dfrac{\partial \mathcal{E}^{123}}{\partial x^n} = -\left(\Gamma^1_{nm}\mathcal{E}^{m23} + \Gamma^2_{nm}\mathcal{E}^{1m3} + \Gamma^3_{nm}\mathcal{E}^{12m}\right)$. Collecting non-zero terms, we have $\dfrac{\partial \mathcal{E}^{123}}{\partial x^n} = -\left(\Gamma^1_{n1}\mathcal{E}^{123} + \Gamma^2_{n2}\mathcal{E}^{123} + \Gamma^3_{n3}\mathcal{E}^{123}\right) = -\mathcal{E}^{123}\Gamma^i_{ni}$, summation applies on i. Recalling that $\mathcal{E}^{123} = 1/\mathcal{J}$ and $\Gamma^i_{ni} = \Gamma^i_{in}$, we get $\dfrac{\partial}{\partial x^n}\left(\dfrac{1}{\mathcal{J}}\right) = -\dfrac{1}{\mathcal{J}}\Gamma^i_{in}$ (recall $\mathcal{J}$ is the Jacobian). Performing the differentiation and rearranging terms, we get

$$\Gamma^i_{in} = \frac{1}{\mathcal{J}}\frac{\partial \mathcal{J}}{\partial x^n} \tag{10.26}$$

Note that Equation 10.26 works for arbitrary coordinate systems and it recovers Equation 10.23 when the system is orthogonal.

10.4.1. Example: Christoffel symbols for cylindrical and spherical coordinate systems

In this example, we calculate and derive the relations for Christoffel symbols of the second kind, for cylindrical and spherical coordinate systems. The results are also expressed in terms of unit vectors (i.e., the physical components) for each coordinate system. Readers should note that the results depend on the order of coordinate axes, defined. In other words, attention should be given when reading comparable results from other references in relation to their defined corresponding coordinates axes.

Cylindrical coordinates

From the results of Example 8.1, for a cylindrical coordinate system (r,θ,z) (note the order of the coordinates defined here, see Figure 8.2) we can write the non-zero terms of the metric tensor, as $g_{rr} = h_r^2 = 1$, $g_{\theta\theta} = h_\theta^2 = r^2$, and $g_{zz} = h_z^2 = 1$. Therefore, the contravariant metric tensor reads $g^{rr} = 1$, $g^{\theta\theta} = r^{-2}$, and $g^{zz} = 1$. Using Equations 10.20 and 10.25 and considering that indices i, j, k, m are various permutations of cylindrical variables r,θ,z we get the Christoffel symbol of the second kind as

$$\begin{cases} \Gamma^r = \begin{bmatrix} \Gamma^r_{rr} & \Gamma^r_{r\theta} & \Gamma^r_{rz} \\ \Gamma^r_{\theta r} & \Gamma^r_{\theta\theta} & \Gamma^r_{\theta z} \\ \Gamma^r_{zr} & \Gamma^r_{z\theta} & \Gamma^r_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \Gamma^\theta = \begin{bmatrix} \Gamma^\theta_{rr} & \Gamma^\theta_{r\theta} & \Gamma^\theta_{rz} \\ \Gamma^\theta_{\theta r} & \Gamma^\theta_{\theta\theta} & \Gamma^\theta_{\theta z} \\ \Gamma^\theta_{zr} & \Gamma^\theta_{z\theta} & \Gamma^\theta_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \Gamma^z = \begin{bmatrix} \Gamma^z_{rr} & \Gamma^z_{r\theta} & \Gamma^z_{rz} \\ \Gamma^z_{\theta r} & \Gamma^z_{\theta\theta} & \Gamma^z_{\theta z} \\ \Gamma^z_{zr} & \Gamma^z_{z\theta} & \Gamma^z_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{cases} \qquad 10.27$$

For example, $\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^\theta_{r\theta} = \frac{1}{h_\theta}\frac{\partial(h_\theta)}{\partial r} = \frac{1}{r}\frac{\partial r}{\partial r} = 1/r$ and $\Gamma^r_{\theta\theta} = -\frac{h_\theta}{h_r^2}\frac{\partial(h_\theta)}{\partial r} = -r\frac{\partial r}{\partial r} = -r$. Recall that the symbols, given by Equation 10.27 are defined based on the covariant basis vectors, $\vec{e}_i$ (see Equation 10.5), or $\frac{\partial \vec{e}_r}{\partial \theta} = \underbrace{(1/r)}_{\Gamma^\theta_{r\theta}}\vec{e}_\theta$ and $\frac{\partial \vec{e}_\theta}{\partial \theta} = \underbrace{(-r)}_{\Gamma^r_{\theta\theta}}\vec{e}_r$. It is useful to write the Christoffel symbols in terms of the physical components of the vectors involved. This is done by using Equation 3.8 (i.e. $\vec{e}_i = h_i\vec{e}(i)$), or $\frac{\partial(r\vec{e}(\theta))}{\partial \theta} = -r\vec{e}(r)$, hence $\frac{\partial(\vec{e}(\theta))}{\partial \theta} = -\vec{e}(r)$, or the Christoffel symbol based on unit vector reads $\hat{\Gamma}^r_{\theta\theta} = -1$ (hatted to distinguish them from the relation given in Equation 10.27). Similarly, for $\frac{\partial \vec{e}_r}{\partial \theta} = \underbrace{(1/r)}_{\Gamma^\theta_{r\theta}}\vec{e}_\theta$ we can write $\frac{\partial(\vec{e}(r))}{\partial \theta} = \frac{1}{r}(r\vec{e}(\theta))$, hence $\frac{\partial(\vec{e}(r))}{\partial \theta} = \vec{e}(\theta)$, or the Christoffel symbol based on unit vector reads $\hat{\Gamma}^\theta_{r\theta} = 1$.

We could also calculate the Christoffel symbols $\hat{\Gamma}^\theta_{r\theta}$ and $\hat{\Gamma}^r_{\theta\theta}$, using the unit vectors or the physical components, as given in Example 8.1. Writing these relations in vector form, we have $\vec{e}(r) = \begin{Bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{Bmatrix}$, $\vec{e}(\theta) = \begin{Bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{Bmatrix}$, and $\vec{e}(z)=\begin{Bmatrix}0\\0\\1\end{Bmatrix}$. Therefore, the derivatives are; $\dfrac{\partial\vec{e}(r)}{\partial r}=\dfrac{\partial\vec{e}(\theta)}{\partial r}=\dfrac{\partial\vec{e}(z)}{\partial r}=0$,

$\dfrac{\partial\vec{e}(r)}{\partial\theta}=\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix}=\vec{e}(\theta)$, $\quad\dfrac{\partial\vec{e}(\theta)}{\partial\theta}=\begin{Bmatrix}-\cos\theta\\ -\sin\theta\\ 0\end{Bmatrix}=-\vec{e}(r)$, $\dfrac{\partial\vec{e}(z)}{\partial\theta}=0$, $\quad\dfrac{\partial\vec{e}(r)}{\partial z}=$ $\dfrac{\partial\vec{e}(\theta)}{\partial z}=\dfrac{\partial\vec{e}(z)}{\partial z}=0$. Therefore, the non-zero Christoffel symbols calculated based on the unit vectors are $\hat{\Gamma}^{\theta}_{r\theta}=1$ and $\hat{\Gamma}^{r}_{\theta\theta}=-1$. The final results can be summarized as

$$\begin{cases}\hat{\Gamma}^{r}=\begin{bmatrix}\hat{\Gamma}^{r}_{rr} & \hat{\Gamma}^{r}_{r\theta} & \hat{\Gamma}^{r}_{rz}\\ \hat{\Gamma}^{r}_{\theta r} & \hat{\Gamma}^{r}_{\theta\theta} & \hat{\Gamma}^{r}_{\theta z}\\ \hat{\Gamma}^{r}_{zr} & \hat{\Gamma}^{r}_{z\theta} & \hat{\Gamma}^{r}_{zz}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&-1&0\\0&0&0\end{bmatrix}\\ \hat{\Gamma}^{\theta}=\begin{bmatrix}\hat{\Gamma}^{\theta}_{rr} & \hat{\Gamma}^{\theta}_{r\theta} & \hat{\Gamma}^{\theta}_{rz}\\ \hat{\Gamma}^{\theta}_{\theta r} & \hat{\Gamma}^{\theta}_{\theta\theta} & \hat{\Gamma}^{\theta}_{\theta z}\\ \hat{\Gamma}^{\theta}_{zr} & \hat{\Gamma}^{\theta}_{z\theta} & \hat{\Gamma}^{\theta}_{zz}\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}\\ \hat{\Gamma}^{z}=\begin{bmatrix}\hat{\Gamma}^{z}_{rr} & \hat{\Gamma}^{z}_{r\theta} & \hat{\Gamma}^{z}_{rz}\\ \hat{\Gamma}^{z}_{\theta r} & \hat{\Gamma}^{z}_{\theta\theta} & \hat{\Gamma}^{z}_{\theta z}\\ \hat{\Gamma}^{z}_{zr} & \hat{\Gamma}^{z}_{z\theta} & \hat{\Gamma}^{z}_{zz}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\end{cases} \qquad 10.28$$

Spherical coordinates

From the results of Example 8.2, for a spherical system (r,φ,θ), (note the order of the coordinates defined here, see Figure 8.3) we can write the non-zero terms of the metric tensor as $g_{rr}=h_r^2=1$, $g_{\varphi\varphi}=h_\varphi^2=r^2$, and $g_{\theta\theta}=h_\theta^2=r^2\sin^2\varphi$. Therefore, the contravariant metric tensor reads $g^{rr}=1$, $g^{\varphi\varphi}=1/r^2$, and $g^{\theta\theta}=1/\left(r^2\sin^2\varphi\right)$. Using Equations 10.20 and 10.25 and considering that indices i,j,k,m are various permutations of spherical variables r,φ,θ we get the Christoffel symbol of the second kind as

$$\begin{cases} \Gamma^r = \begin{bmatrix} \Gamma^r_{rr} & \Gamma^r_{r\varphi} & \Gamma^r_{r\theta} \\ \Gamma^r_{\varphi r} & \Gamma^r_{\varphi\varphi} & \Gamma^r_{\varphi\theta} \\ \Gamma^r_{\theta r} & \Gamma^r_{\theta\varphi} & \Gamma^r_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\varphi \end{bmatrix} \\ \Gamma^\varphi = \begin{bmatrix} \Gamma^\varphi_{rr} & \Gamma^\varphi_{r\varphi} & \Gamma^\varphi_{r\theta} \\ \Gamma^\varphi_{\varphi r} & \Gamma^\varphi_{\varphi\varphi} & \Gamma^\varphi_{\varphi\theta} \\ \Gamma^\varphi_{\theta r} & \Gamma^\varphi_{\theta\varphi} & \Gamma^\varphi_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & -\dfrac{\sin 2\varphi}{2} \end{bmatrix} \\ \Gamma^\theta = \begin{bmatrix} \Gamma^\theta_{rr} & \Gamma^\theta_{r\varphi} & \Gamma^\theta_{r\theta} \\ \Gamma^\theta_{\varphi r} & \Gamma^\theta_{\varphi\varphi} & \Gamma^\theta_{\varphi\theta} \\ \Gamma^\theta_{\theta r} & \Gamma^\theta_{\theta\varphi} & \Gamma^\theta_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/r \\ 0 & 0 & \cot\varphi \\ 1/r & \cot\varphi & 0 \end{bmatrix} \end{cases} \qquad 10.29$$

For example, $\Gamma^2_{33} = \Gamma^\varphi_{\theta\theta} = -\frac{h_3}{h_2^2}\frac{\partial h_3}{\partial x^2} = -\frac{r\sin\varphi}{r^2}\frac{\partial(r\sin\varphi)}{\partial\varphi} = -\sin\varphi\cos\varphi = -\frac{\sin 2\varphi}{2}$. Recall that the symbols are defined based on the covariant basis vectors, $\vec{e}_i$ (see Equation 10.5), or $\frac{\partial\vec{e}_\theta}{\partial\theta} = \underbrace{\left(-\frac{\sin 2\varphi}{2}\right)}_{\Gamma^\varphi_{\theta\theta}}\vec{e}_\varphi$.

We can write this relation in terms of the physical components, as $\frac{\partial(r\sin\varphi\,\vec{e}(\theta))}{\partial\theta} = \left(-\frac{\sin 2\varphi}{2}\right)r\vec{e}(\varphi)$, or $\frac{\partial(\vec{e}(\theta))}{\partial\theta} = \underbrace{(-\cos\varphi)}_{\hat{\Gamma}^\varphi_{\theta\theta}}\vec{e}(\varphi)$. Similarly, the other non-zero symbols can be written as; $\Gamma^r_{\varphi\varphi} = -\frac{h_\varphi}{h_r^2}\frac{\partial h_\varphi}{\partial r} = -r\frac{\partial r}{\partial r} = -r$, $\Gamma^r_{\theta\theta} = -\frac{h_\theta}{h_r^2}\frac{\partial h_\theta}{\partial r} = -r\sin\varphi\frac{\partial(r\sin\varphi)}{\partial r} = -r\sin^2\varphi$, $\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{1}{h_\varphi}\frac{\partial h_\varphi}{\partial r} = \frac{1}{r}\frac{\partial r}{\partial r} = \frac{1}{r}$, $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{h_\theta}\frac{\partial h_\theta}{\partial r} = \frac{1}{r\sin\varphi}\frac{\partial(r\sin\varphi)}{\partial r} = \frac{1}{r}$, and $\Gamma^\theta_{\varphi\theta} = \Gamma^\theta_{\theta\varphi} = \frac{1}{h_\theta}\frac{\partial h_\theta}{\partial\varphi} = \frac{1}{r\sin\varphi}\frac{\partial(r\sin\varphi)}{\partial\varphi} = \cot\varphi$. In terms of physical components of the corresponding vectors, we have $\frac{\partial\vec{e}_\varphi}{\partial\varphi} = \underbrace{(-r)}_{\Gamma^r_{\varphi\varphi}}\vec{e}_r$ or $\frac{\partial(r\vec{e}(\varphi))}{\partial\varphi} = -r\vec{e}(r) = -\vec{e}(r)$, hence $\hat{\Gamma}^r_{\varphi\varphi} = -1$. Also, $\frac{\partial\vec{e}_\theta}{\partial\theta} = \underbrace{(-r\sin^2\varphi)}_{\Gamma^r_{\theta\theta}}\vec{e}_r$

or $\dfrac{\partial\left(r\sin\varphi\,\vec{e}(\theta)\right)}{\partial\theta} = -r\sin^2\varphi\,\vec{e}(r) = \underbrace{\left(-\sin\varphi\right)}_{\hat{\Gamma}^{r}_{\theta\theta}}\vec{e}(r)$, hence $\hat{\Gamma}^{r}_{\theta\theta} = -\sin\varphi$.

Also, $\dfrac{\partial\vec{e}_r}{\partial\varphi} = \underbrace{\left(1/r\right)}_{\Gamma^{\varphi}_{r\varphi}}\vec{e}_\varphi$ or $\dfrac{\partial\vec{e}(r)}{\partial\varphi} = \left(1/r\right)r\vec{e}(\varphi) = \vec{e}(\varphi)$, hence $\hat{\Gamma}^{\varphi}_{r\varphi} = 1$. Also,

$\dfrac{\partial\vec{e}_r}{\partial\theta} = \underbrace{\left(1/r\right)}_{\Gamma^{\theta}_{r\theta}}\vec{e}_\theta$ or $\dfrac{\partial\vec{e}(r)}{\partial\theta} = \left(1/r\right)\left(r\sin\varphi\,\vec{e}(\theta)\right) = \underbrace{\sin\varphi}_{\hat{\Gamma}^{\theta}_{r\theta}}\,\vec{e}(\theta)$, hence $\hat{\Gamma}^{\theta}_{r\theta} = \sin\varphi$.

Also, $\dfrac{\partial\vec{e}_\varphi}{\partial\theta} = \underbrace{\left(\cot\varphi\right)}_{\Gamma^{\theta}_{\varphi\theta}}\vec{e}_\theta$ or $\dfrac{\partial\left(r\vec{e}(\varphi)\right)}{\partial\theta} = \cot\varphi\left(r\sin\varphi\,\vec{e}(\theta)\right) = \underbrace{\cos\varphi}_{\hat{\Gamma}^{\theta}_{\varphi\theta}}\,\vec{e}(\theta)$, hence $\hat{\Gamma}^{\theta}_{\varphi\theta} = \cos\varphi$.

The Christoffel symbols can be obtained directly using the unit vectors as well. From the results of Example 8.1, writing the physical components relations in vector form, we have $\vec{e}(r) = \begin{Bmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{Bmatrix}$, $\vec{e}(\varphi) = \begin{Bmatrix} \cos\varphi\cos\theta \\ \cos\varphi\sin\theta \\ -\sin\varphi \end{Bmatrix}$, and $\vec{e}(\theta) = \begin{Bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{Bmatrix}$. Therefore, the derivatives are; $\dfrac{\partial\vec{e}(r)}{\partial r} = \dfrac{\partial\vec{e}(\varphi)}{\partial r} = \dfrac{\partial\vec{e}(\theta)}{\partial r} = 0$,

$$\frac{\partial\vec{e}(r)}{\partial\varphi} = \begin{Bmatrix} \cos\varphi\cos\theta \\ \cos\varphi\sin\theta \\ -\sin\varphi \end{Bmatrix} = \vec{e}(\varphi), \qquad \frac{\partial\vec{e}(\varphi)}{\partial\varphi} = \begin{Bmatrix} -\sin\varphi\cos\theta \\ -\sin\varphi\sin\theta \\ -\cos\varphi \end{Bmatrix} = -\vec{e}(r), \frac{\partial\vec{e}(\theta)}{\partial\varphi} = 0,$$

$$\frac{\partial\vec{e}(r)}{\partial\theta} = \begin{Bmatrix} -\sin\varphi\sin\theta \\ \sin\varphi\cos\theta \\ 0 \end{Bmatrix} = \vec{e}(\theta)\sin\varphi, \qquad \frac{\partial\vec{e}(\varphi)}{\partial\theta} = \begin{Bmatrix} -\cos\varphi\sin\theta \\ \cos\varphi\cos\theta \\ 0 \end{Bmatrix} = \vec{e}(\theta)\cos\varphi,$$

$\dfrac{\partial\vec{e}(\theta)}{\partial\theta} = \begin{Bmatrix} -\cos\theta \\ -\sin\theta \\ 0 \end{Bmatrix} = -\vec{e}(r)\sin\varphi - \vec{e}(\varphi)\cos\varphi$. Therefore, the non-zero Christoffel symbols calculated based on the unit vectors are

$$\begin{cases} \hat{\Gamma}^r = \begin{bmatrix} \hat{\Gamma}^r_{rr} & \hat{\Gamma}^r_{r\varphi} & \hat{\Gamma}^r_{r\theta} \\ \hat{\Gamma}^r_{\varphi r} & \hat{\Gamma}^r_{\varphi\varphi} & \hat{\Gamma}^r_{\varphi\theta} \\ \hat{\Gamma}^r_{\theta r} & \hat{\Gamma}^r_{\theta\varphi} & \hat{\Gamma}^r_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\sin\varphi \end{bmatrix} \\ \hat{\Gamma}^\varphi = \begin{bmatrix} \hat{\Gamma}^\varphi_{rr} & \hat{\Gamma}^\varphi_{r\varphi} & \hat{\Gamma}^\varphi_{r\theta} \\ \hat{\Gamma}^\varphi_{\varphi r} & \hat{\Gamma}^\varphi_{\varphi\varphi} & \hat{\Gamma}^\varphi_{\varphi\theta} \\ \hat{\Gamma}^\varphi_{\theta r} & \hat{\Gamma}^\varphi_{\theta\varphi} & \hat{\Gamma}^\varphi_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\cos\varphi \end{bmatrix} \\ \hat{\Gamma}^\theta = \begin{bmatrix} \hat{\Gamma}^\theta_{rr} & \hat{\Gamma}^\theta_{r\varphi} & \hat{\Gamma}^\theta_{r\theta} \\ \hat{\Gamma}^\theta_{\varphi r} & \hat{\Gamma}^\theta_{\varphi\varphi} & \hat{\Gamma}^\theta_{\varphi\theta} \\ \hat{\Gamma}^\theta_{\theta r} & \hat{\Gamma}^\theta_{\theta\varphi} & \hat{\Gamma}^\theta_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \sin\varphi \\ 0 & 0 & \cos\varphi \\ \sin\varphi & \cos\varphi & 0 \end{bmatrix} \end{cases} \qquad 10.30$$

Readers can use the Christoffel symbols of the 2nd kind calculated in this example and further calculate the Christoffel symbols of the 1st kind using Equation 10.19.

CHAPTER 11

DERIVATIVE FORMS—CURL, DIVERGENCE, LAPLACIAN

In the governing equations for physical phenomena we usually have terms which contain various forms of gradient operator, including the gradient vector itself—for example, when the gradient vector operates as cross product or dot product with another tensor quantity. The cross-product is called the *curl*, $\vec{\nabla} \times \vec{A}$ and the dot-product is the *divergence*, $\vec{\nabla} \cdot \vec{A}$. For example, in fluid mechanics the curl of the velocity vector is the vorticity vector and its divergence is null, for incompressible fluids.

As mentioned, one of the objectives of tensor analysis is to provide a tool to write down the governing equations in coordinate-independent forms while the covariancy or contravariancy of tensor quantities are correctly implemented in these equations. In the following sections, we derive the coordinate-independent relations for curl, divergence, Laplacian, and biharmonic operators of tensors for an arbitrary coordinate system.

11.1 CURL OPERATIONS ON TENSORS

To form the curl operator expression, we first consider it for a vector in 3D coordinate space and then extend it to higher order dimension. In Section 9, we gave a list of guidelines for forming expressions for tensors. Following these guidelines and using Equation 9.6, we can write the covariant component of the curl of a vector as $C_i = \left(\vec{\nabla} \times \vec{A}\right)_i = \mathcal{E}_{ijk} \nabla^j A^k$. We know that $\mathcal{E}_{ijk}$ is a tensor (see Section 9). But the expression $\nabla^j A^k$ does not transform like a tensor, due to extra terms appearing when transformed to another arbitrary coordinate system [4], [7]. However, if we use the results obtained in Section 10 and replace $\nabla^j A^k$ with the tensor $A^{k\,j}_{,\,\cdot}$ (i.e., the contravariant derivative of the contravariant component) then we get the proper expression as (see Equation 10.8)

$$C_i = \mathcal{E}_{ijk} A^{k\,j}_{,\,\cdot} \qquad 11.1$$

Similarly, the contravariant component of the curl is written as, using Equations 9.6 and 10.12,

$$C^i = \mathcal{E}^{ijk} A_{k,j} \qquad 11.2$$

Both expressions given by Equations 11.1 and 11.2 are tensors, since terms are involved are tensors as well as both reduce to the Cartesian forms when written in the Cartesian coordinate system (i.e., $C_i = e_{ijk} \dfrac{\partial A_k}{\partial y^j}$ because the covariant and contravariant components are identical, and Jacobian is unity).

To have a more practical and explicit relation for the curl, we write Equation 11.2 in detail or $C^i = \mathcal{E}^{ijk} A_{k,j} = \mathcal{E}^{ijk} \dfrac{\partial A_k}{\partial x^j} - \mathcal{E}^{ijk} \Gamma^n_{jk} A_n$ examining the last term, on the R.H.S, using the symmetry property of the Christoffel symbol (i.e., $\Gamma^n_{jk} = \Gamma^n_{kj}$) and the anti-symmetry of the permutation symbol (i.e., $\mathcal{E}^{ijk} = -\mathcal{E}^{ikj}$) yields, $\mathcal{E}^{ijk} \Gamma^n_{jk} A_n = -\mathcal{E}^{ikj} \Gamma^n_{kj} A_n$. But after interchanging $j \leftrightarrow k$, dummy indices, only in the term on the R.H.S of the latter expression we get $-\mathcal{E}^{ikj} \Gamma^n_{kj} A_n = -\mathcal{E}^{ijk} \Gamma^n_{jk} A_n$. After comparing this result with the original expression (i.e., $-\mathcal{E}^{ijk} \Gamma^n_{jk} A_n = \mathcal{E}^{ijk} \Gamma^n_{jk} A_n$) we can conclude that $\mathcal{E}^{ijk} \Gamma^n_{jk} A_n = 0$. Therefore, Equation 11.2 can be written as

$$C^i = \left(\vec{\nabla} \times \vec{A} \right)^i = \frac{e_{ijk}}{\mathcal{J}} \left(\frac{\partial A_k}{\partial x^j} \right) \qquad 11.3$$

The curl vector is then written as $\vec{C} = C^i \vec{e}_i$.

Similarly, we write Equation 11.1 in detail, alongside Equation 10.8, or $C_i = \mathcal{E}_{ijk} A^{k\,j}_{,\,\cdot} = \mathcal{E}_{ijk} g^{jn} \dfrac{\partial A^k}{\partial x^n} + \mathcal{E}_{ijk} g^{jn} \Gamma^k_{nm} A^m$. Examining the last term on the R.H.S, it turns out that it doesn't vanish. Therefore, we have

$$C_i = \left(\vec{\nabla}\times\vec{A}\right)_i = \mathcal{J}g^{jn}e_{ijk}\left(\frac{\partial A^k}{\partial x^n}+\Gamma^k_{nm}A^m\right) \qquad 11.4$$

Note that the curl operation result is a tensor of the same rank as the original quantities. As in our example $\vec{C}$ is a vector, like $\vec{\nabla}$ and $\vec{A}$.

Now, we extend the discussion and find the relation for tensors of higher rank. For simplicity of writing the expressions, without losing generality, we consider a mixed tensor of second rank A^i_j. The invariant quantity is $A = A^i_j\vec{e}_i\vec{e}^j$ and the curl reads $\vec{\nabla}\times A = \vec{\nabla}\times\left(A^i_j\vec{e}_i\vec{e}^j\right)$. The gradient operator just differentiates all the terms in front of it but the vector part of $\vec{\nabla}$ can form a cross-product with either $\vec{e}_i$ or $\vec{e}^j$, both are possible and legitimate operations. One can form, then two forms of the curl of the quantity A. In general, we can have N number of forms for the curl of a tensor of order N. For our example here, we have two forms:

1. We consider the case for which the cross-product operation occurring with $\vec{e}_i$. Therefore, the contravariant component of A^i_j should be involved with the permutation tensor indices. We can write, using Equation 11.1, the component of curl of A as $\left(\vec{\nabla}\times\vec{e}_i\right)A^i_j\vec{e}^j$. Careful attention should be given to pick up the right indices to form the curl expression. Since we picked the $\vec{e}_i$ for cross-product operation with gradient, the covariant permutation tensor should be used, say $\mathcal{E}_{kmn}$. Therefore, the index m corresponds to the contravariant differentiation and index n summed up with the contravariant component of the tensor A. The index k, is then the covariant component of the result or the curl tensor. But the rank of the result should be the same as the original quantity A^i_j (i.e., rank two), hence the free covariant index j remains intact and we end up with a doubly-covariant tensor, or

$$C_{kj} = \mathcal{E}_{kmn}A^{n\,m}_{j,\,\cdot} = \mathcal{J}g^{mp}e_{kmn}\left(\frac{\partial A^n_j}{\partial x^p}+\Gamma^n_{pq}A^q_j\right) \qquad 11.5$$

2. We consider the case for which the cross-product operation occurs with $\vec{e}^j$. Therefore, the covariant component of A^i_j should be involved with the permutation tensor. We can write, using Equation 11.1, the component of curl of A as $\left(\vec{\nabla}\times\vec{e}^j\right)A^i_j\vec{e}_i$. Careful attentions should be given to pick up the right indices to form the curl expression. Since we picked the $\vec{e}^j$ for cross-product operation with gradient, the contravariant

permutation tensor should be used, say $\mathcal{E}^{kmn}$. Therefore, the index m corresponds to the covariant differentiation and index n is summed up with the covariant component of the tensor A. The index k, is then the contravariant component of the result or the curl. But the rank of the result should be the same as the original quantity A^i_j (i.e., rank two), hence the free contravariant index i remains intact and we end up with a doubly contravariant tensor, or

$$C^{ki} = \mathcal{E}^{kmn} A^i_{n,m} = \frac{e_{kmn}}{\mathcal{J}} \left(\frac{\partial A^i_n}{\partial x^m} \right) \qquad 11.6$$

A similar operation can be used to write down the curl of a tensor of rank N, which has N possible outcomes.

11.2 PHYSICAL COMPONENTS OF THE CURL OF TENSORS—3D ORTHOGONAL SYSTEMS

In many applications, we consider orthogonal systems (i.e., $g_{ij} = 0$, $i \neq j$). In this section, we derive expressions for the physical components of the curl of a vector $\vec{A}$ in 3D orthogonal systems. Recalling from previous sections (see Sections 3 and 8), we can write $\vec{A} = A^i \vec{e}_i = A_i \vec{e}^{\,i} = A(i)\vec{e}(i)$ and $h_i = |\vec{e}_i|$. Using $\vec{e}_i \cdot \vec{e}^{\,j} = \delta_i^j$, for orthogonal systems we have $\vec{e}_i \cdot \vec{e}^{\,i} = 1$ which means that the covariant basis vector $\vec{e}_i$ and contravariant basis vector $\vec{e}^{\,i}$ both point in the same direction. For example, $\vec{e}^{\,1}$ and $\vec{e}_1$ are along the same line and point in the same direction. Therefore,

$$\left|\vec{e}^{\,i}\right| = 1 / h_i \qquad 11.7$$

Substituting $\vec{e}_i = h_i \vec{e}(i)$ in $A(i)\vec{e}(i) = A^i \vec{e}_i$ and $\vec{e}^{\,i} = \vec{e}(i)/h_i$ in $A(i)\vec{e}(i) = A_i \vec{e}^{\,i}$ gives

$$A(i) = h_i A^i = A_i / h_i \qquad \textit{no sum on } i \qquad 11.8$$

Or $A(1) = h_1 A^1 = A_1 / h_1$, $A(2) = h_2 A^2 = A_2 / h_2$, $A(3) = h_3 A^3 = A_3 / h_3$.

Now we would like to write the physical components of the curl of vector $\vec{A}$. To do this we multiply Equation 11.3 by h_i, or $\left(\vec{\nabla}\times\vec{A}\right)^i h_i = h_i C^i = h_i \dfrac{e_{ijk}}{\mathcal{J}}\left(\dfrac{\partial A_k}{\partial x^j}\right)$. But $h_i C^i = C(i)$ and $A_k = h_k A(k)$ (see Equation 11.8). Therefore, we will get the physical component $C(i)$ of the curl of vector $\vec{A}$ in terms of its physical component $A(k)$, as

$$C(i) = h_i \frac{e_{ijk}}{\mathcal{J}} \frac{\partial\left(h_k A(k)\right)}{\partial x^j} \qquad \textit{no sum on i, orthogonal} \qquad 11.9$$

Or, using $\mathcal{J} = h_1 h_2 h_3$ for orthogonal systems, we have

$$\begin{cases} \left(\vec{\nabla}\times\vec{A}\right)(1) = C(1) = \dfrac{1}{h_2 h_3}\left[\dfrac{\partial\left(h_3 A(3)\right)}{\partial x^2} - \dfrac{\partial\left(h_2 A(2)\right)}{\partial x^3}\right] \\ \left(\vec{\nabla}\times\vec{A}\right)(2) = C(2) = \dfrac{1}{h_1 h_3}\left[\dfrac{\partial\left(h_1 A(1)\right)}{\partial x^3} - \dfrac{\partial\left(h_3 A(3)\right)}{\partial x^1}\right] \\ \left(\vec{\nabla}\times\vec{A}\right)(3) = C(3) = \dfrac{1}{h_1 h_2}\left[\dfrac{\partial\left(h_2 A(2)\right)}{\partial x^1} - \dfrac{\partial\left(h_1 A(1)\right)}{\partial x^2}\right] \end{cases} \qquad 11.10$$

It can be concluded that for a Cartesian system we recover the familiar expressions, or $\left(\vec{\nabla}\times\vec{A}\right)_1 = C_1 = \left(\dfrac{\partial A_z}{\partial Y} - \dfrac{\partial A_y}{\partial Z}\right)$, $\left(\vec{\nabla}\times\vec{A}\right)_2 = C_2 = \left(\dfrac{\partial A_X}{\partial Z} - \dfrac{\partial A_Z}{\partial X}\right)$, and $\left(\vec{\nabla}\times\vec{A}\right)_3 = C_3 = \left(\dfrac{\partial A_Y}{\partial X} - \dfrac{\partial A_X}{\partial Y}\right)$, since $h_1 = h_2 = h_3 = 1$.

11.3 DIVERGENCE OPERATION ON TENSORS

To form the divergence operator expression, we first consider it for a vector in 3D coordinate space and then extend the results to higher order dimension. In Section 9, we gave a list of guidelines for forming expressions. Following these guidelines and using Equation 9.2, we can write the divergence (using the covariant component of the gradient) as

$$\vec{\nabla}\cdot\vec{A} = A^i_{,i} \qquad 11.11$$

which reduces to proper expression in Cartesian coordinates and shows that the divergence of a vector results in a scalar, or in general divergence operation reduces the rank of the tensor, on which it is operating, by one. To further simplify Equation 11.11, we write it explicitly (see Equation 10.6) as $A^i_{,i} = \frac{\partial A^i}{\partial x^i} + \Gamma^i_{in} A^n$ and plug in for Γ^i_{in} from Equation 10.26 (after interchanging dummy indices $i \leftrightarrow n$), to get $A^i_{,i} = \frac{\partial A^i}{\partial x^i} + \frac{A^i}{\mathcal{J}}\frac{\partial \mathcal{J}}{\partial x^i}$. After substituting for $A^i_{,i}$ into Equation 11.11 and performing some manipulations, we get

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{\mathcal{J}} \frac{\partial \left(\mathcal{J} A^i \right)}{\partial x^i} \tag{11.12}$$

In addition, Equation 11.11 can be written in terms of the covariant component of the vector using g^{ij}, or $\vec{\nabla} \cdot \vec{A} = \left(g^{ij} A_j \right)_{,i}$, after expanding gives $\vec{\nabla} \cdot \vec{A} = g^{ij} \left(A_j \right)_{,i} + A_j \underbrace{\left(g^{ij} \right)_{,i}}_{=0} = g^{ij} A_{j,i}$ or after interchanging $i \leftrightarrow j$ and using the symmetry of metric tensor, we get

$$\vec{\nabla} \cdot \vec{A} = g^{ij} A_{i,j} \tag{11.13}$$

Now we extend the discussion by considering a tensor of higher rank, for example and without losing generality a tensor of second rank $A = A^i_j \vec{e}_i \vec{e}^{\,j}$. The divergence of A is $\vec{\nabla} \cdot A = \vec{\nabla} \cdot \left(A^i_j \vec{e}_i \vec{e}^{\,j} \right)$ in which we have choices between $\vec{e}_i$ or $\vec{e}^{\,j}$ for the gradient vector performing a dot-product operation. Therefore, we have two cases for this example:

1. $\vec{\nabla}$ dot products with $\vec{e}_i$: In this case we get $\vec{\nabla} \cdot A = \vec{\nabla} \cdot \vec{e}_i \left(A^i_j \vec{e}^{\,j} \right)$. Note that the gradient differentiates all terms in front of it but as a vector onlydotted with $\vec{e}_i$. In other words, the contravariant index i is involved in the dot-product operation. Using Equation 11.11 we can write $\vec{\nabla} \cdot A = A^i_{j,i}$ which is a tensor of rank one, as expected since divergence reduces the rank of A^i_j by one. But using Equation 11.12, we have $A^i_{j,i} = \frac{\partial A^i_j}{\partial x^i} + \Gamma^i_{im} A^m_j - \Gamma^m_{ij} A^i_m$ in which the first two terms on the R.H.S. gives $\frac{\partial A^i_j}{\partial x^i} + \Gamma^i_{im} A^m_j = \frac{1}{\mathcal{J}} \frac{\partial \left(\mathcal{J} A^i_j \right)}{\partial x^i}$, using Equation 11.12. Finally, we get the result as

$$\vec{\nabla} \cdot A = \vec{\nabla} \cdot \vec{e}_i \left(A^i_j \vec{e}^{\,j} \right) = \frac{1}{\mathcal{J}} \frac{\partial \left(\mathcal{J} A^i_j \right)}{\partial x^i} - \Gamma^m_{ij} A^i_m \tag{11.14}$$

2. $\vec{\nabla}$ dot products with $\vec{e}^{\,j}$: In this case we get $\vec{\nabla}\cdot A=\vec{\nabla}\cdot\vec{e}^{\,j}\left(A^i_j\vec{e}_i\right)$. Note that the gradient differentiates all terms in front of it but as a vector only dotted with $\vec{e}^{\,j}$. In other words, the index j is involved in the dot-product operation. Using Equation 11.11 we can write $\vec{\nabla}\cdot A=g^{kj}A^i_{k,j}$ which is a tensor of rank one, as expected since divergence reduces the rank of A^i_j by one. Therefore, we have

$$\vec{\nabla}\cdot A=\vec{\nabla}\cdot\vec{e}^{\,j}\left(A^i_j\vec{e}_i\right)=A^{i\ \ k}_{k,\ .} \qquad 11.15$$

Using similar operations, we can write the expression for divergence of a tensor of rank N. Both relations given by Equations 11.14 and 11.15 reduce to familiar forms for divergence in a Cartesian coordinate system. For example, for divergence of vector $\vec{A}$ we get $\vec{\nabla}\cdot\vec{A}=A_{i,i}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$.

11.4 LAPLACIAN OPERATIONS ON TENSORS

A Laplacian operator is the result of a gradient operator forming a dot-product with itself, or divergence of gradient, $\nabla^2=\left(\vec{\nabla}\cdot\vec{\nabla}\right)$. Performing Laplacian on a scalar ψ and using Equation 11.12, we can write $\nabla^2\psi=\vec{\nabla}\cdot\left(\vec{\nabla}\psi\right)=\left(\nabla\psi\right)^i_{,i}$, or $\nabla^2\psi=\frac{1}{\mathcal{J}}\frac{\partial\left(\mathcal{J}\nabla^i\psi\right)}{\partial x^i}$ where $\nabla^i\psi=g^{ij}\frac{\partial\psi}{\partial x^j}$. Therefore,

$$\nabla^2\psi=\frac{1}{\mathcal{J}}\frac{\partial}{\partial x^i}\left(\mathcal{J}g^{ij}\frac{\partial\psi}{\partial x^j}\right) \qquad 11.16$$

Since ∇^2 is a tensor of rank zero, then $\nabla^2\psi$ is a scalar, as well. In general, the rank is determined by the quantity that Laplacian is operating on.

We extend the discussion to find the Laplacian of a vector, for example $\vec{A}=A^i\vec{e}_i$. Since for Laplacian the dot-product operation is performed between the two gradients involved, we don't have many choices, as was the case for the divergence operator, and hence the formulation is more definite in terms of the results. Using Equations 11.13 and 11.16, we can write the contravariant component of the gradient of vector $\vec{A}$ as $g^{nm}A^i_{,n}$ and the components of the Laplacian as $\left(g^{nm}A^i_{,n}\right)_{,m}$. Note that index m indicates the dot-product operation between the contravariant component of the gradient of $\vec{A}$ and the covariant derivative of the second gradient involved. Expanding the last expression, we get $\left(g^{nm}A^i_{,n}\right)_{,m}=g^{nm}_{,m}A^i_{,n}+g^{nm}A^i_{,nm}$. But $g^{nm}_{,m}=0$, since it is a tensor and its value is zero in the Cartesian coordinate system, hence it should be zero for an arbitrary system, as well. Therefore, we have

$$\left(\nabla^2\vec{A}\right)^i = \left(g^{nm}A^i_{,n}\right)_{,m} = g^{nm}A^i_{,nm} \tag{11.17}$$

The term $A^i_{,nm}$ which is the second covariant derivative is a new term, so far. By expanding this term, we get $A^i_{,nm} = \left(A^i_{,n}\right)_{,m} = \dfrac{\partial\left(A^i_{,n}\right)}{\partial x^m} + \Gamma^i_{mj}A^j_{,n} - \Gamma^j_{mn}A^i_{,j}$. But $A^i_{,n} = \dfrac{\partial\left(A^i\right)}{\partial x^n}A^i + \Gamma^i_{nk}A^k$ and after substituting and rearranging similar terms, we get

$$\begin{aligned} A^i_{,nm} = {} & \frac{\partial^2 A^i}{\partial x^m \partial x^n} + A^k\frac{\partial\Gamma^i_{nk}}{\partial x^m} + \Gamma^i_{nk}\frac{\partial A^k}{\partial x^m} + \Gamma^i_{mj}\frac{\partial A^i}{\partial x^j} \\ & - \Gamma^j_{mn}\frac{\partial A^i}{\partial x^j} + \Gamma^i_{mj}\Gamma^j_{nk}A^k - \Gamma^j_{mn}\Gamma^i_{jk}A^k \end{aligned} \tag{11.18}$$

Similar expressions can be written for higher order tensors, using hints from Equation 11.18 for writing the terms involving Christoffel symbols with appropriate signs. The higher the rank of $\vec{A}$ the more complicated the related expression for $\nabla^2\vec{A}$ becomes.

For an orthogonal coordinate system, Equation 11.17 simplifies, since $g^{nm} = 0$ for $m \neq n$, or (for $N = 3$)

$$\left(\nabla^2\vec{A}\right)^i = g^{11}A^i_{,11} + g^{22}A^i_{,22} + g^{33}A^i_{,33} \quad (orthogonal)$$

11.5 BIHARMONIC OPERATIONS ON TENSORS

A biharmonic operator is the result of a Laplacian operator operating on itself, or $\nabla^4 = \nabla^2\left(\nabla^2\right)$. Performing the operation on a scalar ψ and using Equation 11.12, we can write

$$\nabla^4\psi = \frac{1}{\mathcal{J}}\frac{\partial}{\partial x^k}\left\{\mathcal{J}g^{kl}\frac{\partial}{\partial x^l}\underbrace{\left[\frac{1}{\mathcal{J}}\frac{\partial}{\partial x^i}\left(\mathcal{J}g^{ij}\frac{\partial\psi}{\partial x^j}\right)\right]}_{\nabla^2\psi}\right\} \tag{11.19}$$

When a biharmonic operates on a vector, the result is a vector as well. For example, for $\vec{A} = A^i \vec{e}_i$, we can write $\nabla^4 \vec{A} = B^i \vec{e}_i$. Since the result is a vector we can use Equation 11.17 to write

$$B^i = g^{lm} g^{jk} A^i_{,jklm} \tag{11.20}$$

The quantity $A^i_{,jklm}$ is a lengthy expression and can be written in detail [2], [4], [7] using hints taken from Equation 11.18. In principle, one can extend the discussion to find operators like $\nabla^{2n} = \underbrace{\nabla^2 \nabla^2 \cdots \nabla^2}_{n\ times}$. In practice we encounter mostly up to the level of the biharmonic operator in governing equations.

11.6 PHYSICAL COMPONENTS OF THE LAPLACIAN OF A VECTOR—3D ORTHOGONAL SYSTEMS

In this section we derive expressions for physical components of the Laplacian of a vector. Relations for physical components are useful since they are the magnitude of the components using unit vectors as the scale of measurement (see Section 6).

We start with Equation 11.18 and write it for a vector, like $\vec{A} = A(i)\vec{e}(i)$ or $\nabla^2 \vec{A} = \frac{1}{\mathcal{J}} \frac{\partial}{\partial x^i}\left(\mathcal{J} g^{ij} \frac{\partial \vec{A}}{\partial x^j} \right)$. In an orthogonal system, we have $g^{ij} = 0$ (when $i \neq j$) and $g^{ii} = 1/h_i^2$. Therefore, after substitution into the expression, we will have $\nabla^2 \vec{A} = \frac{1}{\mathcal{J}} \frac{\partial}{\partial x^i}\left(\frac{\mathcal{J}}{h_i^2} \frac{\partial \left[A(i)\vec{e}(i) \right]}{\partial x^i} \right)$. Performing the differentiations along with using related relations for Christoffel symbols and after some manipulations, we get the physical j-component as [7]

$$\begin{aligned}
\nabla^2 \vec{A}\Big|_{(j)} = {} & \nabla^2 A(j) + 2\left[\frac{1}{h_i h_j^2} \frac{\partial h_j}{\partial x^i} \frac{\partial A(i)}{\partial x^j} - \frac{1}{h_j h_i^2} \frac{\partial h_i}{\partial x^j} \frac{\partial A(i)}{\partial x^i} \right] \\
& + \left[\frac{A(i)}{h_i h_j^3} \frac{\partial h_j}{\partial x^j} \frac{\partial h_j}{\partial x^i} + \frac{A(i)}{h_j h_i^3} \frac{\partial h_i}{\partial x^i} \frac{\partial h_i}{\partial x^j} \right] - \left[\frac{A(j)}{h_i^2 h_j^2} \frac{\partial h_j}{\partial x^i} \frac{\partial h_j}{\partial x^i} + \frac{A(k)}{h_k h_j h_i^2} \frac{\partial h_i}{\partial x^k} \frac{\partial h_i}{\partial x^j} \right] \\
& + \left[\frac{A(i)}{\mathcal{J}} \frac{\partial}{\partial x^j}\left(\frac{\mathcal{J}}{h_i h_j^2} \frac{\partial h_j}{\partial x^i} \right) - \frac{A(i)}{\mathcal{J}} \frac{\partial}{\partial x^i}\left(\frac{\mathcal{J}}{h_j h_i^2} \frac{\partial h_i}{\partial x^j} \right) \right]
\end{aligned} \tag{11.21}$$

In Equation 11.21, summation is done on indices i and k but not on index j, since it is the free index. To show the application of Equation 11.21, we present two examples to calculate the components of Laplacian of a vector in cylindrical and spherical coordinate systems. Note that the Laplacian of a vector is a vector as well, which has three components.

11.6.1 Example: Physical components of the Laplacian of a vector—cylindrical systems

Consider a cylindrical coordinate system $\left(x^1,x^2,x^3\right)\equiv(r,\theta,z)$ where r is the radial distance, θ the azimuthal angle, and z the vertical coordinate, w.r.t. the Cartesian coordinates $\left(y^1,y^2,y^3\right)\equiv(X,Y,Z)$. The functional relations are $\begin{cases} X = r\cos\theta \\ Y = r\sin\theta \\ Z = z \end{cases}$. Find the Laplacian expressions for a scalar and the physical components of a vector in this system. Let $A(1)=A_r$, $A(2)=A_\theta$, and $A(3)=A_z$.

Solution:

To find the covariant basis vectors, we use $\vec{e}_i = \dfrac{\partial y^j}{\partial x^i}\vec{E}_j$. Therefore, we get $\vec{e}_1 = \dfrac{\partial X}{\partial r}\vec{E}_1 + \dfrac{\partial Y}{\partial r}\vec{E}_2 + \dfrac{\partial Z}{\partial r}\vec{E}_3 = \cos\theta\,\vec{E}_1 + \sin\theta\,\vec{E}_2$. Similarly, $\vec{e}_2 = \dfrac{\partial X}{\partial \theta}\vec{E}_1 + \dfrac{\partial Y}{\partial \theta}\vec{E}_2 + \dfrac{\partial Z}{\partial \theta}\vec{E}_3 = -r\sin\theta\,\vec{E}_1 + r\cos\theta\,\vec{E}_2$. And $\vec{e}_3 = \dfrac{\partial X}{\partial z}\vec{E}_1 + \dfrac{\partial Y}{\partial z}\vec{E}_2 + \dfrac{\partial Z}{\partial z}\vec{E}_3 = \vec{E}_3$. Therefore, we have

$$\begin{cases} \vec{e}_r = \cos\theta\,\vec{E}_1 + \sin\theta\,\vec{E}_2 \\ \vec{e}_\theta = -r\sin\theta\,\vec{E}_1 + r\cos\theta\,\vec{E}_2 \\ \vec{e}_z = \vec{E}_3 \end{cases}$$

The scale factors are the magnitudes of the basis vectors. Hence $h_1 = h_r = \sqrt{\vec{e}_r\cdot\vec{e}_r} = 1$, $h_2 = h_\theta = \sqrt{\vec{e}_\theta\cdot\vec{e}_\theta} = r$, and $h_3 = h_z = \sqrt{\vec{e}_z\cdot\vec{e}_z} = 1$. The unit vectors $\vec{e}(i) = \dfrac{\vec{e}_i}{h_i}$, are

$$\begin{cases} \vec{e}(r) = \cos\theta\,\vec{E}_1 + \sin\theta\,\vec{E}_2 \\ \vec{e}(\theta) = -\sin\theta\,\vec{E}_1 + \cos\theta\,\vec{E}_2 \\ \vec{e}(\varphi) = \vec{E}_3 \end{cases}$$

The Jacobin is $\mathcal{J} = h_r h_\theta h_z = r$ and $g^{ii} = h_i^{-2}$, or $g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Now using Equation 11.16 for orthogonal system i.e. $\nabla^2\psi = \frac{1}{\mathcal{J}}\frac{\partial}{\partial x^i}\left(\mathcal{J}g^{ii}\frac{\partial\psi}{\partial x^i}\right)$, since $g^{ij} = 0$ for $i \neq j$, and substituting for corresponding values we will get $\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$. In other words, the Laplacian operator in cylindrical coordinate system reads

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}\right) + \frac{\partial^2}{\partial z^2} \qquad 11.22$$

Similarly, using Equation 11.21 we get the physical components of the Laplacian for the vector $\vec{A}$, as

$$\begin{cases} \nabla^2\vec{A}\Big|_r = \nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{A_r}{r^2} \\ \nabla^2\vec{A}\Big|_\theta = \nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{A_\theta}{r^2} \\ \nabla^2\vec{A}\Big|_z = \nabla^2 A_z \end{cases} \qquad 11.23$$

Where the operator ∇^2 is given by Equation 11.22 and A_r, A_θ, A_z represent the physical components of $\vec{A}$.

11.6.2 Example: Physical components of the Laplacian of a vector—spherical systems

Consider a spherical coordinate system $(x^1, x^2, x^3) \equiv (r, \varphi, \theta)$ where r is the radial distance, φ the polar and θ the azimuthal angle, w.r.t. the Cartesian coordinates $(y^1, y^2, y^3) \equiv (X, Y, Z)$. The functional relations are

$\begin{cases} X = r\sin\varphi\cos\theta \\ Y = r\sin\varphi\sin\theta \\ Z = r\cos\varphi \end{cases}$. Find the Laplacian expressions for a scalar and the physical components of a vector in this system. Let $A(1) = A_r$, $A(2) = A_\varphi$, and $A(3) = A_\theta$.

Solution:

To find the covariant basis vectors, we use $\vec{e}_i = \frac{\partial y^j}{\partial x^i}\vec{E}_j$. Therefore, we get $\vec{e}_1 = \frac{\partial X}{\partial r}\vec{E}_1 + \frac{\partial Y}{\partial r}\vec{E}_2 + \frac{\partial Z}{\partial r}\vec{E}_3 = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3$. Similarly, $\vec{e}_2 = \frac{\partial X}{\partial \varphi}\vec{E}_1 + \frac{\partial Y}{\partial \varphi}\vec{E}_2 + \frac{\partial Z}{\partial \varphi}\vec{E}_3 = r\cos\varphi\cos\theta\vec{E}_1 + r\cos\varphi\sin\theta\vec{E}_2 - r\sin\varphi\vec{E}_3$. And $\vec{e}_3 = \frac{\partial X}{\partial \theta}\vec{E}_1 + \frac{\partial Y}{\partial \theta}\vec{E}_2 + \frac{\partial Z}{\partial \theta}\vec{E}_3 = -r\sin\varphi\sin\theta\vec{E}_1 + r\sin\varphi\cos\theta\vec{E}_2$. Therefore, we have

$$\begin{cases} \vec{e}_r = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3 \\ \vec{e}_\varphi = r\cos\varphi\cos\theta\vec{E}_1 + r\cos\varphi\sin\theta\vec{E}_2 - r\sin\varphi\vec{E}_3 \\ \vec{e}_\theta = -r\sin\varphi\sin\theta\vec{E}_1 + r\sin\varphi\cos\theta\vec{E}_2 \end{cases}$$

The scale factors are the magnitudes of the basis vectors. Hence $h_1 = h_r = \sqrt{\vec{e}_r \cdot \vec{e}_r} = 1$, $h_2 = h_\varphi = \sqrt{\vec{e}_\varphi \cdot \vec{e}_\varphi} = r$, and $h_3 = h_\theta = \sqrt{\vec{e}_\theta \cdot \vec{e}_\theta} = r\sin\varphi$. The unit vectors $\vec{e}(i) = \frac{\vec{e}_i}{h_i}$, are

$$\begin{cases} \vec{e}(r) = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3 \\ \vec{e}(\varphi) = \cos\varphi\cos\theta\vec{E}_1 + \cos\varphi\sin\theta\vec{E}_2 - \sin\varphi\vec{E}_3 \\ \vec{e}(\varphi) = -\sin\theta\vec{E}_1 + \cos\theta\vec{E}_2 \end{cases}$$

The Jacobin is $\mathcal{J} = h_r h_\varphi h_\theta = r^2\sin\varphi$ and $g^{11} = h_i^{-2}$, or $g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/\left(r^2\sin^2\varphi\right) \end{bmatrix}$. Now using Equation 11.16 for orthogonal system i.e. $\nabla^2\psi = \frac{1}{\mathcal{J}}\frac{\partial}{\partial x^i}\left(\mathcal{J}g^{ii}\frac{\partial\psi}{\partial x^i}\right)$, since $g^{ij} = 0 \text{ for } i \neq j$, and substituting for corresponding values we will get $\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\psi}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2\psi}{\partial\theta^2}$. In other words, the Laplacian operator in spherical coordinate system reads

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2} \qquad 11.24$$

Similarly, using Equation 11.21 we get the physical components of the Laplacian for the vector $\vec{A}$, as

$$\begin{cases} \nabla^2\vec{A}\Big|_r = \nabla^2 A_r - \dfrac{2A_r}{r^2} - \dfrac{2}{r^2\sin\varphi}\dfrac{\partial(\sin\varphi\, A_\varphi)}{\partial\varphi} - \dfrac{2}{r^2\sin\varphi}\dfrac{\partial A_\theta}{\partial\theta} \\ \nabla^2\vec{A}\Big|_\varphi = \nabla^2 A_\varphi - \dfrac{A_\varphi}{r^2\sin^2\varphi} + \dfrac{2}{r^2}\dfrac{\partial A_r}{\partial\varphi} - \dfrac{2\cos\varphi}{r^2\sin^2\varphi}\dfrac{\partial A_\theta}{\partial\theta} \\ \nabla^2\vec{A}\Big|_\theta = \nabla^2 A_\theta - \dfrac{A_\theta}{r^2\sin^2\varphi} + \dfrac{2}{r^2\sin\varphi}\dfrac{\partial A_r}{\partial\theta} + \dfrac{2\cos\varphi}{r^2\sin^2\varphi}\dfrac{\partial A_\varphi}{\partial\theta} \end{cases} \qquad 11.25$$

Where the operator ∇^2 is given by Equation 11.22 and A_r, A_φ, A_θ represent the physical components of $\vec{A}$.

In the next section, we focus on relations pertinent to tensor transformations between merely Cartesian coordinate systems.

CHAPTER 12

CARTESIAN TENSOR TRANSFORMATION—ROTATIONS

When transformation of tensors is performed from one Cartesian coordinate system to another one, relations derived in previous sections take simpler forms, called *Cartesian tensors*. As mentioned previously, a Cartesian system consists of three mutually perpendicular flat surfaces for which; the basis vectors are unit vectors, there is no distinction between covariant and contravariant components, and the components of a tensor are the physical components. Therefore, we use subscript indices for all types of components, regardless.

We will consider two Cartesian systems, y_i and y'_i with a common origin—otherwise we can always redefine them to have a common origin with transforming, say, the origin of y'_i to that of y_i. Following the discussion from Section 3, we can write the differential displacement $\overrightarrow{ds}$ in these systems, as

$$\overrightarrow{ds} = dy_i \vec{E}_i = dy'_j \vec{E}'_j \qquad 12.1$$

where $\vec{E}_i$ and $\vec{E}'_j$ are the corresponding unit vectors. However, we have $dy_i = \dfrac{\partial y_i}{\partial y'_j} dy'_j$, and after substituting in Equation 12.1 we get

$\left(\frac{\partial y_i}{\partial y'_j}\vec{E}_i - \vec{E}'_j\right) dy'_j = 0$. Since dy'_j is arbitrary, we conclude $\frac{\partial y_i}{\partial y'_j}\vec{E}_i = \vec{E}'_j$ or, by $i \leftrightarrow j$, we have

$$\vec{E}'_i = \frac{\partial y_j}{\partial y'_i}\vec{E}_j \qquad 12.2$$

Now performing dot-product on both sides of Equation 12.2 with $\vec{E}_k$ gives, $\vec{E}_k \cdot \vec{E}'_i = \frac{\partial y_j}{\partial y'_i}\underbrace{\vec{E}_j \cdot \vec{E}_k}_{\delta_{jk}} = \frac{\partial y_k}{\partial y'_i}$. Hence,

$$\vec{E}_k \cdot \vec{E}'_i = \frac{\partial y_k}{\partial y'_i} \qquad 12.3$$

This means that the cosine of the angle between y_k and y'_i is equal to $\frac{\partial y_k}{\partial y'_i}$, called the *cosine direction*.

Similarly, we can write $\vec{E}_k = \frac{\partial y'_i}{\partial y_k}\vec{E}'_i$, and after dot-product with $\vec{E}'_j$ we get $\vec{E}'_j \cdot \vec{E}_k = \frac{\partial y'_i}{\partial y_k}\underbrace{\vec{E}'_i \cdot \vec{E}'_j}_{\delta_{ij}} = \frac{\partial y'_j}{\partial y_k}$. Hence, interchanging $i \leftrightarrow j$, we get

$$\vec{E}'_i \cdot \vec{E}_k = \frac{\partial y'_i}{\partial y_k} \qquad 12.4$$

This means that the cosine of the angle between y_k and y'_i is equal to $\frac{\partial y'_i}{\partial y_k}$. Therefore, comparing Equations 12.3 and 12.4 and considering the commutativity of the dot-product operation, we will have

$$\frac{\partial y'_i}{\partial y_k} = \frac{\partial y_k}{\partial y'_i} \qquad 12.5$$

Equation 12.5 yields $dy_i = \frac{\partial y_i}{\partial y'_j}\partial y'_j = \frac{\partial y'_j}{\partial y_i} dy'_j$ and $\vec{E}'_i = \frac{\partial y_j}{\partial y'_i}\vec{E}_j = \frac{\partial y'_i}{\partial y_j}\vec{E}_j$, but only in Cartesian coordinate systems. In other words, these relations prove that there is no distinction between covariant and contravariant components in a Cartesian system (see Equations 4.8 and 5.4, for comparison).

12.1 ROTATION MATRIX

Now if we assume that coordinate system y'_i is obtained by rotating coordinates y_j at an angle θ about an axis parallel to a unit vector $\vec{n}=(n_1, n_2, n_3)$, then the quantity $\dfrac{\partial y'_i}{\partial y_j}=R_{ij}$ is a function of θ and $\vec{n}$ and it can be shown that [9], it reads as

$$R_{ij}=n_i n_j+\left(\delta_{ij}-n_i n_j\right)\cos\theta+e_{ijk}n_k\sin\theta \qquad 12.6$$

Equation 12.6 is *Rodrigues' rotation formula*, or the 3D rotation matrix. Readers should note that the Right-Hand-Rule convention applies to the rotation of coordinates with respect to the positive direction of unit vector $\vec{n}$ when using Equation 12.6. For example, for $\vec{n}=(0,0,1)$ and $\theta=\pi/2$, R_{ij} gives the transformation of $x-y$ plane about the positive direction of z-axis for an angle of $90°$.

Now we integrate $dy'_i=\dfrac{\partial y'_i}{\partial y_j}dy_j=R_{ij}dy_j$ which yields, $y'_i=R_{ij}y_j$. Note that the constant of integration is zero due to having a common origin for both systems, and R_{ij} is a constant for given $\vec{n}$ and θ. Similarly, we can rotate y'_i at an angle $(-\theta)$ about the same unit vector $\vec{n}$ to recover the original y_j coordinates. Therefore, we can write $y_i=R^{-1}_{ij}y'_j=R_{ji}y'_j$, by having R^{-1}_{ij} being equal to the inverse or transpose of R_{ij} [4]. Further examining Equation 12.6, we can conclude that R_{ij} for a given angle θ is equal to R_{ji}, or its transpose, calculated for angle $(-\theta)$. This is the result of the property of the last term on the R.H.S of Equation 12.6 when $i\leftrightarrow j$ to get R_{ji}, or $e_{ijk}n_k\sin\theta=-e_{jik}n_k\sin\theta=e_{jik}n_k\sin(-\theta)$. The remaining terms are not affected by this index-change operation.

12.2 EQUIVALENT SINGLE ROTATION: EIGENVALUES AND EIGENVECTORS

It would be useful to write the matrix form of the Cartesian coordinate rotations, as well. For example (in a 3D space with $N=3$), $y'_i=R_{ij}y_j$ can be written as $\begin{Bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{Bmatrix}=\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$, and after using Equation 12.6, for $\vec{n}=(n_1, n_2, n_3)$ and rotation angle θ, we get

$$\begin{Bmatrix} y_1' \\ y_2' \\ y_3' \end{Bmatrix} = \begin{bmatrix} \cos\theta + n_1 n_1 (1-\cos\theta) & n_1 n_2 (1-\cos\theta) + n_3 \sin\theta & n_1 n_3 (1-\cos\theta) - n_2 \sin\theta \\ n_1 n_2 (1-\cos\theta) - n_3 \sin\theta & \cos\theta + n_2 n_2 (1-\cos\theta) & n_2 n_3 (1-\cos\theta) + n_1 \sin\theta \\ n_1 n_3 (1-\cos\theta) + n_2 \sin\theta & n_2 n_3 (1-\cos\theta) - n_1 \sin\theta & \cos\theta + n_3 n_3 (1-\cos\theta) \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \qquad 12.7$$

From Equation 12.7 (or 12.6) we can conclude that the trace of the rotation matrix (i.e., the sum of diagonal elements, R_{ii}) is equal to $1+2\cos\theta$, (note that $n_i n_i = |\vec{n}|^2 = n_1^2 + n_2^2 + n_3^2 = 1$, since $\vec{n}$ is a unit vector), or

$$\theta = \cos^{-1}\left(\frac{R_{ii} - 1}{2}\right) \qquad 12.8$$

Equation 12.8 is a useful relation for finding rotation angle θ, once R_{ij} is known/given. However, readers may ask: if R_{ij} is known, how can one find the corresponding unit vector parallel to the axis of rotation? This question is all the more significant considering that we can have a sequence of rotations to arrive at a final desired orientation of the coordinate axes. For example, if we rotate the original system through a sequence of rotations and arrive at the final system orientation, i.e., $y_i \to \underbrace{y_i^{t1} \to \cdots \to y_i^{tn}}_{sequence} \to y_i'$, then we can apply Equation 12.7 for each sequence and calculate the final rotation matrix. That is $\{y_i'\} = \underbrace{\left[R_{ij}^{tn}\right]\left[R_{ij}^{tn-1}\right]\cdots\left[R_{ij}^{t1}\right]}_{[R_{ij}]}\{y_i\}$. Note that for each rotation in the sequence, the corresponding rotation matrix is pre-multiplied to the previous ones, and the normal vector parallel to the rotation axis is defined based on the current coordinate system at hand in the sequence (see Example 12.2.1)

Now, if we want to replace all intermediate rotations with just one equivalent rotation, Equation 12.8 gives the value of the angle for the desired equivalent single rotation. However, to find the axis of rotation parallel to the unit vector for the equivalent single rotation we need a mathematical procedure/tool. This tool can be obtained by finding the real eigenvector of the given rotation matrix since, through all rotations, only the eigenvectors remain in the same direction [2], [4]. The mathematical procedure is as follows:

We consider a normal unit vector n_i perpendicular to a plane and write the components of the rotation tensor on this plane, or $R_{ij}n_j$. This quantity is a vector whose components are the elements of the rotation tensor on the plane considered. We identify this quantity as $R_{ij}n_j = T_i$. In general, the direction of the vector T_i does not necessarily coincide with the normal vector to the plane, as shown in Figure 12.1.

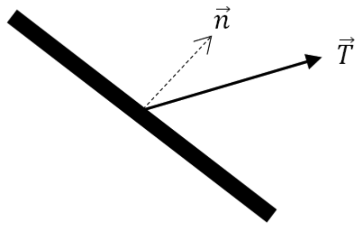


FIGURE 12.1 Rotation matrix component $\vec{T}$, about an arbitrary plane with normal $\vec{n}$.

In other words, we could demand a specific plane and calculate its corresponding normal direction such that vector T_i ends up in the same direction as the normal to the plane considered. Therefore, we can write $T_i n_i = R_{ij}n_j n_i$. This quantity, which is the projection of T_i in the direction of n_i, is a quadratic function of vector n_i. To calculate its extremum value with the constraint that $n_i n_i = 1$ (which means that having the normal vector as a unit vector), we use the Lagrange multiplier method for extremizing the scalar quantity $\mathcal{M} = R_{ij}n_j n_i - \lambda\left(n_i n_i - 1\right)$, where the constant λ is the Lagrange multiplier, or the eigenvalues of R_{ij}. This is achieved by letting $\dfrac{\partial \mathcal{M}}{\partial n_i} = 0$, which leads to (note that the $\dfrac{\partial \mathcal{M}}{\partial \lambda} = 0$ recovers the constraint, $n_i n_i = 1$)

$$\left(R_{ij} - \lambda\delta_{ij}\right)n_j = 0 \qquad 12.9$$

In matrix form, Equation 12.9 can be written as $\begin{bmatrix} R_{11}-\lambda & R_{12} & R_{13} \\ R_{21} & R_{22}-\lambda & \varepsilon_{23} \\ R_{31} & R_{32} & R_{33}-\lambda \end{bmatrix}\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$. This system of equations has a non-trivial (i.e., non-zero) solution, for n_i, if the determinant of the matrix is zero, or

$$\det\left(R_{ij} - \lambda\delta_{ij}\right) = \begin{vmatrix} R_{11}-\lambda & R_{12} & R_{13} \\ R_{21} & R_{22}-\lambda & \varepsilon_{23} \\ R_{31} & R_{32} & R_{33}-\lambda \end{vmatrix} = 0 \qquad 12.10$$

By evaluating the determinant, we get a cubic equation in terms of λ (which has three roots, or eigenvalues) as

$$\lambda^3 - \underbrace{(R_{ii})}_{\mathbb{I}_1}\lambda^2 + \underbrace{\left(\frac{R_{ii}R_{jj} - R_{ij}R_{ji}}{2}\right)}_{\mathbb{I}_2}\lambda - \underbrace{\left|R_{ij}\right|}_{\mathbb{I}_3} = 0 \qquad 12.11$$

Equation 12.11 is known as the *characteristic equation* for matrix R_{ij}, where $\mathbb{I}_1$ is the first *principal scalar invariant,* which is equal to the trace of R_{ij}; or $\mathbb{I}_1 = R_{ii} = \lambda_1 + \lambda_2 + \lambda_3$, $\mathbb{I}_2$ is the second *principal scalar invariant,* which is $\mathbb{I}_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$, or the sum of the diagonal minors; and $\mathbb{I}_3$ is the third *principal scalar invariant,* which is equal to the determinant of R_{ij}, or $\mathbb{I}_3 = e_{ijk}R_{1i}R_{2j}R_{3k} = \lambda_1\lambda_2\lambda_3$.

Equation 12.11 contains several important properties and pieces of information about the system's rotation matrix. Some examples related to the discussion here include:

- For symmetric tensors with real components there exist three real value answers for λ (or λ_1, λ_2, and λ_3). This is not the case for R_{ij}, since it is not symmetric, and we get only one real value λ, [10].
- The answers for λ are the *eigenvalues* (or principal values) of R_{ij}.
- The corresponding n_i, calculated and normalized for each eigenvalue, are the *eigenvectors* (or principal directions) of R_{ij}.
- A given eigenvector when multiplied by a constant real number (positive or negative) it gives a new vector, but all have the same direction parallel to the principal direction.

The method of finding equivalent rotation is very useful in practice for designing the motion of machine parts in robotic applications, such as a robot arm's motion.

12.2.1 Example: Equivalent single rotation to sequential rotations of a Cartesian system

Having a Cartesian system labeled as $y_i = (y_1, y_2, y_3)$ and its transformation to another system labeled as $y'_i = (y'_1, y'_2, y'_3)$ such that y'_1 coincides with y_2, y'_2 with y_3, and y'_3 with y_1, as shown in Figure 12.2, find the overall equivalent single rotation matrix for $y'_i = R_{ij}y_j$, the axis of rotation direction $\vec{n}$ and the angle of rotation θ.

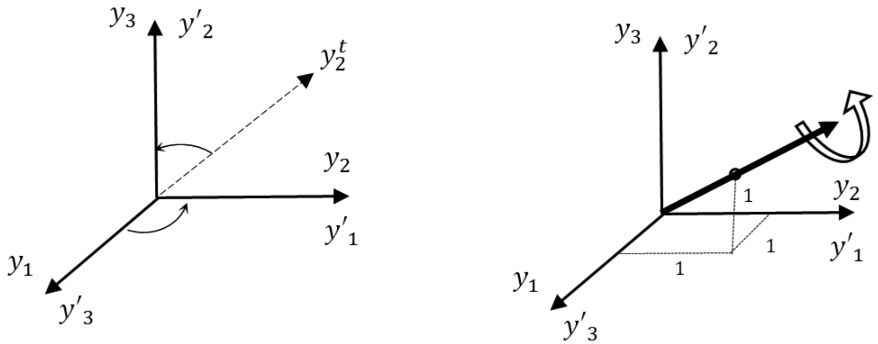


FIGURE 12.2 Sequential rotations of a Cartesian coordinate system (left) and the equivalent single rotation (right).

Solution:

We consider two sequential rotations: 1) rotation about y_3-axis with normal vector $(0,0,1)$ with an angle of 90° (R.H.R. applies) to get the intermediate transformation $(y_1,y_2,y_3)\rightarrow(y'_1,y^t_2,y_3)$, and 2) rotation about new y'_1 with normal vector $(1,0,0)$ with an angle of 90° (R.H.R. applies) to get the second transformation $(y'_1,y^t_2,y_3)\rightarrow(y'_1,y'_2,y'_3)$ (see Figure 12.2). Therefore, using Equation 12.7 and the data given for the first transformation with $\vec{n}=(0,0,1)$ we have $\begin{Bmatrix} y'_1 \\ y^t_2 \\ y_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$. Similarly, for the second rotation with $\vec{n}=(1,0,0)$, we have $\begin{Bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} y'_1 \\ y^t_2 \\ y_3 \end{Bmatrix}$. Note that the second rotation is performed on the current system at hand, which is the result of the first rotation. Therefore, after substitution we can write $\begin{Bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$, or $\begin{Bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$. The final rotation matrix is then $R_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Note that each rotation matrix pre-multiplies the previous one in the sequence of rotations. Having the final rotation matrix, we can calculate the single rotation angle. Using Equation 12.8, we get $\theta = \cos^{-1}\left(\frac{R_{ii}-1}{2}\right) = \cos^{-1}\left(\frac{0-1}{2}\right) = \frac{2\pi}{3}$. The corresponding rotation axis is the eigenvector associated with the real eigenvalue for $R_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. We can write $\left|R_{ij} - \lambda\delta_{ij}\right| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + 1 = 0$. Hence, we get the characteristic equation as $(\lambda - 1)(\lambda^2 + \lambda + 1) = 0$ with its only real root being $\lambda = 1$. Note that we could use Equation 12.11 instead for calculating the eigenvalues. For $\lambda = 1$, we get the eigenvector by solving $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = (1)\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$, or $\begin{cases} n_2 = n_1 \\ n_3 = n_2 \\ n_1 = n_3 \end{cases}$. This system is not determinate but gives $n_1 = n_2 = n_3$. However, the constraint for having $\vec{n}$ as a unit vector gives $n_1^2 + n_2^2 + n_3^2 = 1$, or $3n_1^2 = 1$. Therefore, $n_1 = n_2 = n_3 = \frac{\sqrt{3}}{3}$ and finally we have the equivalent rotation axis or the unit vector $\vec{n} = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.

In conclusion, we can say that instead of two intermediate sequential rotations, as mentioned above, we can rotate the original $y_i = (y_1, y_2, y_3)$ at an angle of $120°$, using right-hand-rule convention, about an axis parallel to the unit vector $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ (see Figure 12.2). Readers should note that for calculating the rotation matrix using Equation 12.7, values of the components of $\vec{n}$ should be those of the unit vector. However, to get the direction of the rotation axis we can multiply the components of the unit vector by a constant—for this example the multiplier is $\sqrt{3}$. As shown in Figure 12.2, the result is a vector, or (1,1,1), which is a vector along the same direction as the unit vector $\vec{n}$. This is consistent with the property of the eigenvalues. We can examine these results by using Equation 12.6 to calculate the single rotation matrix, using $\theta = 120°$ and $\vec{n} = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$, or $R_{11} = \left(\frac{\sqrt{3}}{3}\right)^2 + \left(1 - \left(\frac{\sqrt{3}}{3}\right)^2\right)\cos(120°) = \frac{1}{3} - \frac{1}{3} = 0$, similarly $R_{22} = R_{33} = 0$.

Also, we get $R_{12} = \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\underbrace{\delta_{12}}_{=0} - \left(\frac{\sqrt{3}}{3}\right)^2\right)\cos(120°) + \underbrace{e_{123}}_{=1}\left(\frac{\sqrt{3}}{3}\right)\sin(120°) = 1.$

Similarly, we get $R_{23} = R_{31} = 1$, and the remaining elements $R_{21} = R_{32} = R_{13} = 0$. These results are identical with the one obtained using sequential rotations.

CHAPTER 13

COORDINATE INDEPENDENT GOVERNING EQUATIONS

Reliable mathematical models, also referred to as governing equations, are important tools for engineering analysis. A reliable and validated mathematical model of a physical phenomenon is a set of algebraic relations among quantities and their various derivatives, such as Newton's 2^{nd} law of motion, equilibrium equations for momentum flux, Fourier's law of heat flux, Fick's law of mass flux, Navier-Stokes equations for flow of fluids, Maxwell's electromagnetic equations, etc. These governing equations, along with some fundamental principles (like the 2^{nd} law of thermodynamics, conservations of energy, mass, electric charge, etc.) form the foundation of engineering science and its applications.

The quantities related to any physical phenomenon can be represented by tensors of different ranks stated in the relevant governing equations, such as force and acceleration vectors and mass of a body as a scalar, in Newton's 2^{nd} law of motion; gradient of temperature in Fourier's law; divergence of velocity vector in fluid flow, etc. Sometimes, for analysis and design purposes, we need to have the relevant governing equations written in coordinate systems other than the Cartesian system. For example, for analyzing mechanical stresses in the wall of a cylindrical pressure vessel we prefer to choose the cylindrical polar coordinate system that is a natural fit to the shape of the vessel. This requirement has encouraged scientists and engineers to define various coordinate systems suitable for solving the governing equations in practice. In other words, and in the context of tensor analysis, we would like to use the relations derived and discussed in the previous sections to write down the terms involved in well-known governing equations in engineering in a form that is general enough for application in an arbitrary but well-defined coordinate system.

In this section we derive some new relations mostly used in engineering, in addition to those discussed in the previous sections. We hope that this helps readers in writing down similar coordinate independent terms involved in equations of their choice for their applications.

13.1 THE ACCELERATION VECTOR—CONTRAVARIANT COMPONENTS

Newton's 2nd law is a mathematical model for the motion mechanics of physical objects—specifically, it is the balance of applied forces and the rate of change of momentum, or mass times acceleration. Velocity is the time rate of change of displacement vector $\overrightarrow{ds}$ (see Section 2). Therefore, in an arbitrary coordinate system x^i, we can write the velocity vector $\vec{v}$ as $\vec{v} = \frac{\overrightarrow{ds}}{dt}$ for the corresponding time increment of dt. Using Equation 3.6, we have $\vec{v} = \frac{\overrightarrow{ds}}{dt} = \vec{e}_i \frac{dx^i}{dt}$. Since $\overrightarrow{ds}$ is a vector, hence $\vec{v}$ is a vector and we can write it in terms of its contravariant component v^i, or

$$\vec{v} = v^i \vec{e}_i = \frac{dx^i}{dt} \vec{e}_i \tag{13.1}$$

Equation 13.1 clearly gives the contravariant component of the velocity vector as $v^i = \frac{dx^i}{dt}$. Acceleration vector $\vec{a}$ is the time derivative of the velocity vector, or

$$\vec{a} = \frac{d\vec{v}}{dt} = a^i \vec{e}_i \tag{13.2}$$

Using Equation 13.1, we can write $\frac{d\vec{v}}{dt} = \frac{d\left(v^i \vec{e}_i\right)}{dt} = \vec{e}_i \frac{dv^i}{dt} + v^i \frac{d\vec{e}_i}{dt}$. To relate the time derivative to the space gradient we use chain rule $\frac{d}{dt} = \underbrace{\frac{dx^j}{dt}}_{v^j} \frac{\partial}{\partial x^j}$, hence

$$\frac{d}{dt} = v^j \frac{\partial}{\partial x^j} \qquad 13.3$$

Substituting into Equation 13.2, we get $\vec{a} = \frac{d\vec{v}}{dt} = \vec{e}_i \left(v^j \frac{\partial v^i}{\partial x^j} \right) + v^i \left(v^j \frac{\partial \vec{e}_i}{\partial x^j} \right)$.
But $\frac{\partial \vec{e}_i}{\partial x^j} = \Gamma^k_{ij} \vec{e}_k$, using Equation 10.5. Therefore, $\vec{a} = \vec{e}_i v^j \frac{\partial v^i}{\partial x^j} + v^i v^j \Gamma^k_{ij} \vec{e}_k$ and in the last term we interchange dummy indices $i \leftrightarrow k$ and use the symmetry property of the Christoffel symbol to get

$$\vec{a} = \left(v^j \frac{\partial v^i}{\partial x^j} + v^k v^j \Gamma^i_{jk} \right) \vec{e}_i \qquad 13.4$$

Comparing Equations 13.2 and 13.4, we get the contravariant component of the acceleration vector as

$$a^i = v^j \left(\frac{\partial v^i}{\partial x^j} + v^k \Gamma^i_{jk} \right) \qquad 13.5$$

In addition, we can write $v^i_{,j} = \frac{\partial v^i}{\partial x^j} + v^k \Gamma^i_{jk}$, after using Equation 10.6, to get the compact form of Equation 13.5, as

$$a^i = v^j v^i_{,j} \qquad 13.6$$

Equation 13.6 recovers the familiar expression for acceleration in Cartesian coordinates (i.e., $a^i = v^j \frac{\partial v^i}{\partial x^j}$).
We can also write the covariant component of the acceleration vector using a metric tensor, or $a_k = g_{ik} a^i$ (see Section 8) for an arbitrary coordinate system.

Now, we can write the Newton's 2nd law in a general coordinate independent form. We choose to use the contravariant component of the acceleration vector; hence, for compatibility we must use the contravariant component of the total force vector, $\vec{F} = F^i \vec{e}_i$, as well. Therefore, we can write

$$F^i = m v^j \frac{\partial v^i}{\partial x^j} + m v^k v^j \Gamma^i_{jk} = m v^j v^i_{,j} \qquad 13.7$$

For example, in a 3D coordinate system we get three equations of motions for each dimension, with summation on indices j and k only, as

$$\begin{cases} F^1 = mv^j \dfrac{\partial v^1}{\partial x^j} + mv^k v^j \Gamma^1_{jk} \\ F^2 = mv^j \dfrac{\partial v^2}{\partial x^j} + mv^k v^j \Gamma^2_{jk} \\ F^3 = mv^j \dfrac{\partial v^3}{\partial x^j} + mv^k v^j \Gamma^3_{jk} \end{cases} \quad 13.8$$

Note that in a Cartesian system the last term in Equation 13.7, associated with the Christoffel symbol, is zero and we recover the familiar form of Newton's equation. But when we write this equation in a curvilinear coordinate system, the term associated with the Christoffel symbol is not necessarily equal to zero. This extra term acts like an inertia force and affects the path of moving objects. For example, in a spherical coordinate system, like Earth's geometry, we can write Equation 13.7 as $F^i - mv^k v^j \Gamma^i_{jk} = mv^j \dfrac{\partial v^i}{\partial x^j}$. The left-hand-side can be treated as a new total force, or $\tilde{F}^i = F^i - mv^k v^j \Gamma^i_{jk} = mv^j \dfrac{\partial v^i}{\partial x^j}$. The term $mv^k v^j \Gamma^i_{jk}$ is the *Coriolis force*, F^i_c

$$F^i_c = mv^k v^j \Gamma^i_{jk} \quad 13.9$$

For example, from the point of view of an observer at the north pole of the Earth an object with an initial velocity along a meridian towards the equator will shift to the right, due to the inertia force resulting from the Coriolis force, while to an observer at the south pole it will shift to the left. Or we can say that a force equivalent to the negative of the Coriolis force is acting on the object.

13.2 THE ACCELERATION VECTOR—PHYSICAL COMPONENTS

Physical components of acceleration vector can be obtained using Equation 6.2, or $a(i) = h_i a^i$. Substituting for a^i, Equation 13.5 yields $a(i) = h_i \underbrace{v^j \dfrac{\partial v^i}{\partial x^j}}_{=\frac{dv^i}{dt}} + h_i v^j v^k \Gamma^i_{jk} = h_i \dfrac{dv^i}{dt} + h_i v^j v^k \Gamma^i_{jk}$. In the latter expression, we write the velocity components in terms of their physical components as well to get $a(i)=h_i\frac{d(v(i)/h_i)}{dt}+h_i\frac{v(j)v(k)}{h_jh_k}\Gamma^i_{jk}$. But the first term can be expanded to get $h_i\frac{d(v(i)/h_i)}{dt}=\frac{dv(i)}{dt}+h_iv(i)\frac{d(1/h_i)}{dt}$. Now, we define $\frac{dv(i)}{dt}=\dot{v}(i)$ and use Equation 13.3 to write $h_iv(i)\frac{d(1/h_i)}{dt}=-\frac{1}{h_i}v(i)\frac{d(h_i)}{dt}=-\frac{v(i)v(j)}{h_ih_j}\frac{\partial(h_i)}{\partial x^j}$. Now substituting into the relation for $a(i)$, we obtain

$$a(i)=\dot{v}(i)-\frac{v(i)v(j)}{h_ih_j}\frac{\partial h_i}{\partial x^j}+\frac{h_iv(j)v(k)}{h_jh_k}\Gamma^i_{jk} \qquad 13.10$$

Equation 13.10 gives the physical components of the acceleration vector in curvilinear coordinate systems. Note that for Equation 13.10 summation applies only on indices j and k, but not i.

13.3 THE ACCELERATION VECTOR IN ORTHOGONAL SYSTEMS—PHYSICAL COMPONENTS

A useful application of Equation 13.10 is for orthogonal coordinate systems. As mentioned previously, for an orthogonal system we should have $g_{ij}=g^{ij}=0$ for $i\neq j$ and we can use relations given by Equations 10.23–10.25 for Christoffel symbols. Therefore, expanding Equation 13.10, for example for $i=1$, we get $a(1)=\dot{v}(1)-\frac{v(1)v(j)}{h_1h_j}\frac{\partial h_1}{\partial x^j}+\frac{h_1v(j)v(k)}{h_jh_k}\Gamma^1_{jk}$, which after writing the summations on j and k yields $a(1)=\dot{v}(1)-\frac{v(1)}{h_1}\underbrace{\left[\frac{v(1)}{h_1}\frac{\partial h_1}{\partial x^1}+\frac{v(2)}{h_2}\frac{\partial h_1}{\partial x^2}+\frac{v(3)}{h_3}\frac{\partial h_1}{\partial x^3}\right]}_{\text{sum on } j}$

$$+\frac{h_1v(1)}{h_1}\left[\underbrace{\frac{v(1)}{h_1}\Gamma^1_{11}+\frac{v(2)}{h_2}\Gamma^1_{12}+\frac{v(3)}{h_3}\Gamma^1_{13}}_{\text{sum on } k,\ j=1}\right]+\frac{h_1v(2)}{h_2}\left[\underbrace{\frac{v(1)}{h_1}\Gamma^1_{21}+\frac{v(2)}{h_2}\Gamma^1_{22}+\frac{v(3)}{h_3}\Gamma^1_{23}}_{\text{sum on } k,\ j=2}\right]+$$

$\frac{h_1 v(3)}{h_3}\left[\underbrace{\frac{v(1)}{h_1}\Gamma^1_{31}+\frac{v(2)}{h_2}\Gamma^1_{32}+\frac{v(3)}{h_3}\Gamma^1_{33}}_{sum\ on\ k,\ \ j=3}\right]$. But from Equations 10.23-10.25, we get $\Gamma^1_{11}=\frac{1}{h_1}\frac{\partial h_1}{\partial x^1}$, $\Gamma^1_{12}=\Gamma^1_{21}=\frac{1}{h_1}\frac{\partial h_1}{\partial x^2}$, $\Gamma^1_{13}=\Gamma^1_{31}=\frac{1}{h_1}\frac{\partial h_1}{\partial x^3}$, $\Gamma^1_{22}=-\frac{h_2}{(h_1)^2}\frac{\partial h_2}{\partial x^1}$, and $\Gamma^1_{33}=-\frac{h_3}{(h_1)^2}\frac{\partial h_3}{\partial x^1}$ and $\Gamma^1_{23}=\Gamma^1_{32}=0$. Substituting for the corresponding Christoffel symbols and simplifying, we get $a(1)=\dot{v}(1)+\frac{v(1)v(2)}{h_1h_2}\frac{\partial h_1}{\partial x^2}+\frac{v(1)v(3)}{h_1h_3}\frac{\partial h_1}{\partial x^3}-\frac{v(2)v(2)}{h_1h_2}\frac{\partial h_2}{\partial x^1}-\frac{v(3)v(3)}{h_1h_3}\frac{\partial h_3}{\partial x^1}$. By similar operation we can calculate the second and third components, i.e., $a(2)$ and $a(3)$. Equation 13.11 lists all the physical components of acceleration vector in a 3D orthogonal coordinate system, as

$$\begin{cases} a(1)=\dot{v}(1)+\frac{v(1)v(2)}{h_1h_2}\frac{\partial h_1}{\partial x^2}+\frac{v(1)v(3)}{h_1h_3}\frac{\partial h_1}{\partial x^3}-\frac{v(2)v(2)}{h_1h_2}\frac{\partial h_2}{\partial x^1}-\frac{v(3)v(3)}{h_1h_3}\frac{\partial h_3}{\partial x^1} \\ a(2)=\dot{v}(2)+\frac{v(1)v(2)}{h_1h_2}\frac{\partial h_2}{\partial x^1}+\frac{v(2)v(3)}{h_2h_3}\frac{\partial h_2}{\partial x^3}-\frac{v(1)v(1)}{h_1h_2}\frac{\partial h_1}{\partial x^2}-\frac{v(3)v(3)}{h_2h_3}\frac{\partial h_3}{\partial x^2} \\ a(3)=\dot{v}(3)+\frac{v(1)v(3)}{h_1h_3}\frac{\partial h_3}{\partial x^1}+\frac{v(2)v(3)}{h_2h_3}\frac{\partial h_3}{\partial x^2}-\frac{v(1)v(1)}{h_1h_3}\frac{\partial h_1}{\partial x^3}-\frac{v(2)v(2)}{h_2h_3}\frac{\partial h_2}{\partial x^3} \end{cases} \tag{13.11}$$

Relations given by Equation 13.11 can be written in index form as, summation on index j only,

$$a(i)=\dot{v}(i)+\frac{v(i)v(j)}{h_ih_j}\frac{\partial h_i}{\partial x^j}-\frac{v(j)v(j)}{h_ih_j}\frac{\partial h_j}{\partial x^i} \tag{13.12}$$

In the next section, we use Equation 13.12 to calculate an acceleration vector's physical components in cylindrical and spherical coordinate systems. For a Cartesian system, Equation 13.12 recovers the familiar relation (i.e., $a(i)=\frac{d(v(i))}{dt}$).

13.3.1 Example: Acceleration vector physical components in cylindrical and spherical coordinate systems

From the results obtained in Examples 8.1 and 8.2, we have the scale factors for cylindrical and spherical coordinates as $(h_r, h_\theta, h_z) = (1, r, 1)$ and $(h_r, h_\varphi, h_\theta) = (1, r, r\sin\varphi)$, respectively. Recall that the cylindrical coordinates are $(x^1, x^2, x^3) \equiv (r, \theta, z)$ and those of a spherical system are $(x^1, x^2, x^3) \equiv (r, \varphi, \theta)$. Now, using Equation 13.11 we can write the physical components of the acceleration vector for cylindrical coordinates as $a(1) = \dot{v}(1) - \dfrac{v(2)v(2)}{r}$, $a(2) = \dot{v}(2) + \dfrac{v(1)v(2)}{r}$, and $a(3) = \dot{v}(3)$. In terms of coordinate notation, we can write it as $a_r = \dot{v}_r - \dfrac{v_\theta^2}{r}$, $a_\theta = \dot{v}_\theta + \dfrac{v_r v_\theta}{r}$, and $a_z = \dot{v}_z$. Note that all components are the physical components of the corresponding vectors. Therefore, the acceleration vector is written as

$$\vec{a} = \left(\dot{v}_r - \frac{v_\theta^2}{r}\right)\vec{e}(r) + \left(\dot{v}_\theta + \frac{v_r v_\theta}{r}\right)\vec{e}(\theta) + \dot{v}_z \vec{e}(z)$$

Note that $\vec{e}(i)$ are unit vectors in cylindrical coordinates. We can also write the acceleration vector in terms of Cartesian unit vectors $\vec{E}_i$, by substituting for $\vec{e}(r) = \cos\theta\,\vec{E}_1 + \sin\theta\,\vec{E}_2$, $\vec{e}(\theta) = -\sin\theta\,\vec{E}_1 + \cos\theta\,\vec{E}_2$, and $\vec{e}(z) = \vec{E}_3$ (see Example 8.1), or

$$\begin{aligned}\vec{a} = &\left[\left(\dot{v}_r - \frac{v_\theta^2}{r}\right)\cos\theta - \left(\dot{v}_\theta + \frac{v_r v_\theta}{r}\right)\sin\theta\right]\vec{E}_1 \\ &+ \left[\left(\dot{v}_r - \frac{v_\theta^2}{r}\right)\sin\theta + \left(\dot{v}_\theta + \frac{v_r v_\theta}{r}\right)\cos\theta\right]\vec{E}_2 + \dot{v}_z \vec{E}_3\end{aligned} \qquad 13.13$$

In terms of coordinates themselves, we have

$$\begin{aligned}\vec{a} = &\left[\left(\ddot{r} - r\dot{\theta}^2\right)\cos\theta - \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\sin\theta\right]\vec{E}_1 \\ &+ \left[\left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\cos\theta\right]\vec{E}_2 + \ddot{z}\vec{E}_3\end{aligned}$$

Similarly, for a spherical coordinate system, using Equation 13.11, we can write the physical components of the acceleration vector for spherical coordinates as $a(1) = a_r = \dot{v}_r - \dfrac{v_\varphi^2}{r} - \dfrac{v_\theta^2}{r}$, $a(2) = a_\varphi = \dot{v}_\varphi + \dfrac{v_r v_\varphi}{r} - \dfrac{v_\theta^2 \cos\varphi}{r\sin\varphi}$, and

$a(3) = a_\theta = \dot{v}_\theta + \frac{v_r v_\theta}{r} + \frac{v_\theta v_\varphi \cos\varphi}{r \sin\varphi}$. Note that all components are the physical components of the corresponding vectors. Therefore, the acceleration vector is

$$\vec{a} = \left(\dot{v}_r - \frac{v_\varphi^2 + v_\theta^2}{r} \right)\vec{e}(r) + \left(\dot{v}_\varphi + \frac{v_r v_\varphi}{r} - \frac{v_\theta^2 \cos\varphi}{r \sin\varphi} \right)\vec{e}(\varphi) + \left(\dot{v}_\theta + \frac{v_r v_\theta}{r} + \frac{v_\theta v_\varphi \cos\varphi}{r \sin\varphi} \right)\vec{e}(\theta)$$

Note that $\vec{e}(i)$ are unit vectors in spherical coordinates. In terms of the coordinates themselves, we have

$$\vec{a} = \left[\ddot{r} - r\dot{\varphi}^2 - r\sin^2\varphi\dot{\theta}^2 \right]\vec{e}(r) + \left[r\ddot{\varphi} + 2\dot{r}\dot{\varphi} - r\sin\varphi\cos\varphi\dot{\theta}^2 \right]\vec{e}(\varphi)$$
$$+ \left[r\ddot{\theta}\sin\varphi + 2\dot{r}\dot{\theta}\sin\varphi + 2r\dot{\varphi}\dot{\theta}\cos\varphi \right]\vec{e}(\theta).$$

We can also write the acceleration vector in terms of Cartesian unit vectors $\vec{E}_i$, by substituting for $\vec{e}(r) = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3$, $\vec{e}(\varphi) = \cos\varphi\cos\theta\vec{E}_1 + \cos\varphi\sin\theta\vec{E}_2 - \sin\varphi\vec{E}_3$, and $\vec{e}(\theta) = -\sin\theta\vec{E}_1 + \cos\theta\vec{E}_2$ (see Example 8.1), or

$$\begin{aligned}\vec{a} = &\left[\left(\dot{v}_r - \frac{v_\varphi^2}{r} - \frac{v_\theta^2}{r} \right)\sin\varphi\cos\theta + \left(\dot{v}_\varphi + \frac{v_r v_\varphi}{r} - \frac{v_\theta^2\cos\varphi}{r\sin\varphi} \right)\cos\varphi\cos\theta - \left(\dot{v}_\theta + \frac{v_r v_\theta}{r} + \frac{v_\theta v_\varphi \cos\varphi}{r\sin\varphi} \right)\sin\theta \right]\vec{E}_1 \\ &+ \left[\left(\dot{v}_r - \frac{v_\varphi^2}{r} - \frac{v_\theta^2}{r} \right)\sin\varphi\sin\theta + \left(\dot{v}_\varphi + \frac{v_r v_\varphi}{r} - \frac{v_\theta^2\cos\varphi}{r\sin\varphi} \right)\cos\varphi\sin\theta + \left(\dot{v}_\theta + \frac{v_r v_\theta}{r} + \frac{v_\theta v_\varphi \cos\varphi}{r\sin\varphi} \right)\cos\theta \right]\vec{E}_2 \\ &+ (\cos\varphi - \sin\varphi)\vec{E}_3\end{aligned} \quad 13.14$$

Equations 13.13 and 13.14 are useful relations for calculating acceleration vectors in cylindrical and spherical coordinate systems, respectively. Similar calculations can be performed for arbitrary orthogonal curvilinear systems.

13.4 SUBSTANTIAL TIME DERIVATIVES OF TENSORS

Another form of derivative of tensors appearing in governing equations is the *substantial derivative*—also referred to as the *total* or *convective time derivative*. For a quantity like A—a scalar, vector, or tensor—that varies with space and time, we would like to collect all its derivatives. Let's assume that we have a body of fluid moving in a fixed coordinate system, for example, a Cartesian system. We consider two scenarios: a) we "ride" the fluid and move with it and register its variation with respect to time, and b) we stay at a fixed location in space and register the variation of the fluid passing by with respect to time. In case (a) the changes are due to any variation of the fluid w.r.t time, since we move with it and relative space change is absent/null. In other words, the change w.r.t space is implicitly included. But in case (b) we may have changes w.r.t both time and space for the fluid in motion.

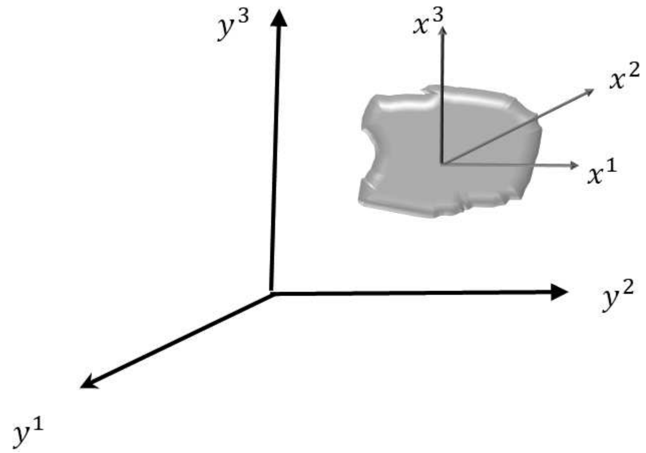


FIGURE 13.1 Loacal coordinates attached to a moving fluid body in a fixed Cartesian coordinate system.

Let's consider function $A = A\left(y^i, t\right)$, where A is the tensor quantity, y^i is the fixed Cartesian coordinate system, and t is time (see Figure 13.1). Here, we assume that the ratio $\frac{\delta y^i}{\delta t} = \upsilon^i$ applies, or the change of space coordinates of the fluid w.r.t time is the same as the local fluid velocity. Now the total time derivative is given by the following limit, $\frac{DA}{Dt} = \lim_{\delta y^i, \delta t \to 0}\left(\frac{\partial A}{\partial t} + \frac{\partial A}{\partial y^i}\frac{\partial y^i}{\partial t}\right)$, or

$$\frac{DA}{Dt} = \frac{\partial A}{\partial t} + \upsilon^i \frac{\partial A}{\partial y^i} \qquad 13.15$$

An immediate extension is to write the total derivative for $A = A^i_j \vec{e}_i \vec{e}^{\,j}$, a mixed tensor of the second rank, in an arbitrary system x^i, or

$$\frac{DA}{Dt} = \frac{\partial\left(A^i_j \vec{e}_i \vec{e}^{\,j}\right)}{\partial t} + \upsilon^k \frac{\partial\left(A^i_j \vec{e}_i \vec{e}^{\,j}\right)}{\partial x^k} \qquad 13.16$$

Using Equation 10.14, we can write the second term on the right-hand-side as $\upsilon^k \dfrac{\partial\left(A^i_j \vec{e}_i \vec{e}^j\right)}{\partial x^k} = \upsilon^k A^i_{j,k} \vec{e}_i \vec{e}^j$. The first term is just equal to $\dfrac{\partial A^i_j}{\partial t}$, if the coordinates x^i are also fixed. Note that if the x^i system is moving, we should consider the changes of the basis vectors with reference to time as well (see Equation 13.16). Therefore, we get the components (mixed contravariant/covariant) of the substantial time derivative of A, as

$$\left(\frac{DA}{Dt}\right)^i_j = \frac{\partial\left(A^i_j\right)}{\partial t} + \upsilon^k A^i_{j,k} \qquad 13.17$$

Note that $A^i_{j,k} = \dfrac{\partial A^i_j}{\partial x^k} + \Gamma^i_{km} A^m_j - \Gamma^n_{kj} A^i_n$.

A specific case is to let A be equal to velocity itself, which results in the total derivative of υ^i, as

$$\left(\frac{D\upsilon}{Dt}\right)^i = \frac{\partial \upsilon^i}{\partial t} + \upsilon^j \upsilon^i_{,j} \qquad 13.18$$

In orthogonal coordinate systems, we can write the physical components the convective time derivative of a vector as

$$\left(\frac{D\vec{A}}{Dt}\right)(i) = \frac{\partial A(i)}{\partial t} + \frac{\upsilon(j)}{h_j}\frac{\partial A(i)}{\partial x^j} + \frac{\upsilon(i)A(j)}{h_i h_j}\frac{\partial h_i}{\partial x^j} - \frac{\upsilon(j)A(j)}{h_i h_j}\frac{\partial h_j}{\partial x^i}, \; no\, sum\, on\, i \qquad 13.19$$

Using Equation 13.19 we can write the total time derivative of vectors like velocity or acceleration. The following example demonstrates these calculations.

13.4.1 Example: Substantial time derivation of acceleration vectors—physical components

From previous examples, we have $(h_r, h_\theta, h_z) = (1, r, 1)$ for cylindrical coordinate systems and $(h_r, h_\varphi, h_\theta) = (1, r, r\sin\varphi)$ for spherical coordinate systems. We can write the physical components of the total derivative of acceleration vector $\vec{a} = (a_r, a_\theta, a_z)$, in cylindrical coordinates, by using Equation 13.19, as

$$\begin{cases} \left(\dfrac{D\vec{a}}{Dt}\right)_r = \dfrac{\partial a_r}{\partial t} + \upsilon_r \dfrac{\partial a_r}{\partial r} + \dfrac{\upsilon_\theta}{r}\dfrac{\partial a_r}{\partial \theta} + \upsilon_z \dfrac{\partial a_r}{\partial z} - \upsilon_\theta \dfrac{a_\theta}{r} \\ \left(\dfrac{D\vec{a}}{Dt}\right)_\theta = \dfrac{\partial a_\theta}{\partial t} + \upsilon_r \dfrac{\partial a_\theta}{\partial r} + \dfrac{\upsilon_\theta}{r}\dfrac{\partial a_\theta}{\partial \theta} + \upsilon_z \dfrac{\partial a_\theta}{\partial z} + \dfrac{\upsilon_\theta a_r}{r} \\ \left(\dfrac{D\vec{a}}{Dt}\right)_z = \dfrac{\partial a_z}{\partial t} + \upsilon_r \dfrac{\partial a_z}{\partial r} + \dfrac{\upsilon_\theta}{r}\dfrac{\partial a_z}{\partial \theta} + \upsilon_z \dfrac{\partial a_z}{\partial z} \end{cases} \qquad 13.20$$

Similarly, for spherical coordinates we get the physical components of the total derivative of acceleration vector as

$$\begin{cases} \left(\dfrac{D\vec{a}}{Dt}\right)_r = \dfrac{\partial a_r}{\partial t} + \upsilon_r \dfrac{\partial a_r}{\partial r} + \dfrac{\upsilon_\varphi}{r}\dfrac{\partial a_r}{\partial \varphi} + \dfrac{\upsilon_\theta}{r\sin\varphi}\dfrac{\partial a_r}{\partial \theta} - \upsilon_\varphi \dfrac{a_\varphi}{r} - \upsilon_\theta \dfrac{a_\theta}{r} \\ \left(\dfrac{D\vec{a}}{Dt}\right)_\varphi = \dfrac{\partial a_\varphi}{\partial t} + \upsilon_r \dfrac{\partial a_\varphi}{\partial r} + \dfrac{\upsilon_\varphi}{r}\dfrac{\partial a_\varphi}{\partial \varphi} + \dfrac{\upsilon_\theta}{r\sin\varphi}\dfrac{\partial a_\varphi}{\partial \theta} + \dfrac{\upsilon_\varphi a_r}{r} - \dfrac{\upsilon_\theta a_\theta \cos\varphi}{r\sin\varphi} \\ \left(\dfrac{D\vec{a}}{Dt}\right)_\theta = \dfrac{\partial a_\theta}{\partial t} + \upsilon_r \dfrac{\partial a_\theta}{\partial r} + \dfrac{\upsilon_\varphi}{r}\dfrac{\partial a_\theta}{\partial \varphi} + \dfrac{\upsilon_\theta}{r\sin\varphi}\dfrac{\partial a_\theta}{\partial \theta} + \dfrac{\upsilon_\theta a_r}{r} + \dfrac{\upsilon_\theta a_\varphi \cos\varphi}{r\sin\varphi} \end{cases} \qquad 13.21$$

Note that in these relations (i.e., Equations 13.20 and 13.21) all components are the physical components of the corresponding vectors.

13.5 CONSERVATION EQUATIONS—COORDINATE INDEPENDENT FORMS

A fundamental equation for the transformation of a tensor quantity per unit volume, Ψ in an arbitrary control volume with a velocity field $\vec{V}$, can be obtained using the Gauss divergence theorem and the law of the principle of conservation [11]. An integral equation defining the time rate and divergence of convective transformation balanced with the rate of production/destruction Ω of the quantity leads to the differential relation given by Equation 13.22, as

$$\frac{\partial \Psi}{\partial t} + \left(\Psi V^i\right)_{,i} = \Omega \qquad 13.22$$

The second term can be written as $\left(\Psi V^i\right)_{,i} = V^i \Psi_{,i} + \Psi V^i_{\ ,i}$ since covariant differentiation follows the differentiation product rule. Therefore, L.H.S of Equation 13.22 can be written as $\underbrace{\frac{\partial\Psi}{\partial t}+V^i\Psi_{,i}}_{=\frac{D\Psi}{Dt}}+\Psi V^i{}_{,i}=\frac{D\Psi}{Dt}+\Psi V^i{}_{,i}$. In vector form, this equation reads

$$\frac{D\Psi}{Dt}+\Psi\vec{\nabla}\cdot\vec{V}=\frac{D\Psi}{Dt}+\Psi V^i{}_{,i}=\Omega \tag{13.23}$$

Equation 13.22 (or equally Equation 13.23) can be considered the equation of motion for the control volume or, in general, the medium that the quantity Ψ transports in it. The source term is associated with the quantity in question: for example, the mass, momentum, energy, electric charge, etc.

Now we consider mass conservation, or let $\Psi=\rho$ where ρ is mass per unit volume or density of the quantity in motion. After substitution in Equation 13.23, we get $\frac{D\rho}{Dt}+\rho V^i{}_{,i}=\Omega$. If mass source is zero, then we get the continuity equation, as

$$\frac{D\rho}{Dt}+\rho V^i{}_{,i}=0 \tag{13.24}$$

Note that $\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+V^i\rho_{,i}$. Some specific cases can be observed:

- $\rho=\rho(t)$ only, i.e., we have an inhomogeneous quantity whose density changes with time. In this case the continuity equation reads as $\frac{\partial\rho}{\partial t}+V^i\underbrace{(\rho_{,i})}_{=0}+\rho V^i{}_{,i}=0$, or $\frac{\partial\rho}{\partial t}+\rho V^i{}_{,i}=0$.
- $\rho=\rho(x^i)$ only, i.e., we have an inhomogeneous quantity whose density changes from location to location in the continuum but remains constant with time at any given location. In this case the continuity equation reads as $\underbrace{\left(\frac{\partial\rho}{\partial t}\right)}_{=0}+V^i\rho_{,i}+\rho V^i{}_{,i}=0$, or $(\rho V^i)=0$.
- ρ is a constant, i.e., we have a homogeneous quantity with constant density at all locations and times in the continuum. In this case the continuity equation reads as $\underbrace{\left(\frac{D\rho}{Dt}\right)}_{=0}+\rho V^i{}_{,i}=0$, or $V^i{}_{,i}=0$.

Now we consider momentum conservation, or let $\Psi = \rho\vec{V}$, i.e., momentum per unit volume of the quantity in motion. In this case Ψ is a vector quantity. Substituting in Equation 13.22, the i^{th} contravariant component reads

$$\frac{\partial\left(\rho V^i\right)}{\partial t}+\left(\rho V^i V^j\right)_{,j}=\Omega^i \qquad 13.25$$

Equation 13.25 is Newton's 2nd law written in the coordinate independent form for a medium, for example a fluid moving in the continuum. The L.H.S is the inertia force and the R.H.S is the applied force on the material, per unit volume. Considering the material as a fluid, then the applied force could be, in general, a combination of hydrostatic, viscous, gravitational, electromagnetic, etc. forces.

CHAPTER 14

COLLECTION OF RELATIONS FOR SELECTED COORDINATE SYSTEMS

In this section we provide a list of some commonly used coordinate systems as well as relations for an arbitrary orthogonal curvilinear coordinate system. We categorize the content based on the coordinate systems. All coordinates are considered in a 3D Euclidean space. Cartesian coordinate systems are fixed references consisting of three flat planes. The curvilinear coordinates may include one or more curved coordinate surfaces.

14.1 CARTESIAN COORDINATE SYSTEM

We use $\left(\vec{e}(1),\vec{e}(2),\vec{e}(3)\right)=\left(\vec{E}_x,\vec{E}_y,\vec{E}_z\right)$, $\left(A(1),A(2),A(3)\right)=\left(A_x,A_y,A_z\right)$ symbols for physical components. Note that in a Cartesian system, all contravariant, covariant, and physical components of a tensor quantity are identical, hence all indices are shown as subscripts.

TABLE 14.1 Relations for tensors and their related derivatives in Cartesian coordinate systems.

Coordinates	$(x, y, z) \equiv (X, Y, Z)$
Basis/Unit vectors, $\vec{E}_i$	$(\vec{E}_x, \vec{E}_y, \vec{E}_z) \equiv (\vec{i}, \vec{j}, \vec{k})$, orthogonal
Scale factors, h_i	$(h_x, h_y, h_z) = (1,1,1)$
Metric tensors, g_{ij}	$g_{xx} = g_{yy} = g_{zz} = 1$, the rest are null
Jacobian	$\mathcal{J} = 1$
Unit volume	$d\mathbb{V} = dxdydz$
Line element and magnitude	$\overrightarrow{dS} = dx\vec{E}_x + dy\vec{E}_y + dz\vec{E}_z$ $\left\|\overrightarrow{dS}\right\| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$
Vector components	$\vec{A} = (A_x, A_y, A_z) = A_x\vec{E}_x + A_y\vec{E}_y + A_z\vec{E}_z$
Christoffel symbols	null
Dot-product (two vectors)	$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$
Cross-product (two vectors)	$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\vec{E}_x + (A_z B_x - A_x B_z)\vec{E}_y + (A_x B_y - A_y B_x)\vec{E}_z$
Gradient vector	$\vec{\nabla} = \vec{E}_x \frac{\partial}{\partial x} + \vec{E}_y \frac{\partial}{\partial y} + \vec{E}_z \frac{\partial}{\partial z}$
Gradient of a scalar, Ψ	$\vec{\nabla}\Psi = \vec{E}_x \frac{\partial \Psi}{\partial x} + \vec{E}_y \frac{\partial \Psi}{\partial y} + \vec{E}_z \frac{\partial \Psi}{\partial z}$
Gradient of a vector, $\vec{A}$	$\vec{\nabla}\vec{A} = \begin{bmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial x} & \frac{\partial A_z}{\partial x} \\ \frac{\partial A_x}{\partial y} & \frac{\partial A_y}{\partial y} & \frac{\partial A_z}{\partial y} \\ \frac{\partial A_x}{\partial z} & \frac{\partial A_y}{\partial z} & \frac{\partial A_z}{\partial z} \end{bmatrix}$
Curl of a vector, $\vec{A}$	$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{E}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{E}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{E}_z$
Divergence of a vector, $\vec{A}$	$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

(continued)

Laplacian of a scalar, Ψ	$\nabla^2\Psi = \vec{\nabla}\cdot\left(\vec{\nabla}\Psi\right) = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}$
Laplacian of a vector, $\vec{A}$	$\nabla^2\vec{A} =$ $\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\vec{E}_x + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\vec{E}_y$ $+\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\vec{E}_z$
Biharmonic operator	$\nabla^4 = \left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4}\right) + 2\left(\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^2\partial z^2} + \frac{\partial^4}{\partial x^2\partial z^2}\right)$

14.2 CYLINDRICAL COORDINATE SYSTEMS

We use $\left(\vec{e}(1),\vec{e}(2),\vec{e}(3)\right) = \left(\vec{e}(r),\vec{e}(\theta),\vec{e}(z)\right)$, $\vec{A} = \left(A(1),A(2),A(3)\right) = \left(A(r),A(\theta),A(z)\right)$ to designate the unit vectors $\vec{e}(i)$ and physical components of vector $\vec{A}$. Most of the expressions are written in terms of physical components. For Christoffel symbols, see Equation 10.28.

TABLE 14.2 Relations for quantities and their related derivatives in cylindrical coordinate systems.

Coordinates (orthogonal)	$\left(x^1,x^2,x^3\right) \equiv \left(r,\theta,z\right)$, θ is the azimuth angle		
Coordinate functions	$x = r\cos\theta$	$y = r\sin\theta$	$z = z$
Coordinate surfaces	$x^2 + y^2 = r^2$, cylinders	$y/x = \tan\theta$, planes	$z = constant$, planes
Basis vectors, $\vec{e}_i$, covariant	$\left(\vec{e}_r,\vec{e}_\theta,\vec{e}_z\right) = \left(\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix},\begin{Bmatrix}-r\sin\theta\\ r\cos\theta\\ 0\end{Bmatrix},\begin{Bmatrix}0\\ 0\\ 1\end{Bmatrix}\right)$		
Basis vectors, $\vec{e}^i$, contravariant	$\left(\vec{e}^r,\vec{e}^\theta,\vec{e}^z\right) = \left(\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix},\begin{Bmatrix}-\sin\theta/r\\ \cos\theta/r\\ 0\end{Bmatrix},\begin{Bmatrix}0\\ 0\\ 1\end{Bmatrix}\right)$		
Scale factors, h_i	$\left(h_r,h_\theta,h_z\right) = \left(1,r,1\right)$		

(continued)

Unit vectors, physical components	$\left(\vec{e}(r),\vec{e}(\theta),\vec{e}(z)\right)=\left(\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix},\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix},\begin{Bmatrix}0\\ 0\\ 1\end{Bmatrix}\right)$
Metric tensors, g_{ij} and g^{ij}	$g_{rr}=g_{zz}=\left(g^{rr}\right)^{-1}=\left(g^{zz}\right)^{-1}=1, g_{\theta\theta}=\left(g^{\theta\theta}\right)^{-1}=r^2$, the rest are null-system is orthogonal
Jacobian	$\mathcal{J}=r$
Unit volume	$d\mathbb{V}=rdrd\theta dz$
Line element and magnitude	$\overrightarrow{dS}=dr\vec{e}(r)+rd\theta\vec{e}(\theta)+dz\vec{e}(z)$ | $\left\lvert\overrightarrow{dS}\right\rvert=\sqrt{(dr)^2+(rd\theta)^2+(dz)^2}$
Vector components	$\vec{A}=A(r)\vec{e}(r)+A(\theta)\vec{e}(\theta)+A(z)\vec{e}(z)$
Christoffel symbols, based on Unit vectors	$\hat{\Gamma}^{r}_{\theta\theta}=-1,\ \hat{\Gamma}^{\theta}_{r\theta}=\hat{\Gamma}^{\theta}_{\theta r}=1$, the rest are null
Christoffel symbols, based on covariant Basis vectors	$\Gamma^{r}_{\theta\theta}=-r,\ \Gamma^{\theta}_{r\theta}=\Gamma^{\theta}_{\theta r}=1/r$, the rest are null
Dot-product (two vectors)	$\vec{A}\cdot\vec{B}=A(r)B(r)+A(\theta)B(\theta)+A(z)B(z)$
Cross-product (two vectors)	$\vec{A}\times\vec{B}=\left[A(\theta)B(z)-A(z)B(\theta)\right]\vec{e}(r)+\left[A(z)B(r)-A(r)B(z)\right]\vec{e}(\theta)+\left[A(r)B(\theta)-A(\theta)B(r)\right]\vec{e}(z)$
Gradient vector	$\vec{\nabla}=\vec{e}_r\dfrac{\partial}{\partial r}+\dfrac{1}{r}\vec{e}_\theta\dfrac{\partial}{\partial\theta}+\vec{e}_z\dfrac{\partial}{\partial z}$
Gradient of a scalar, Ψ	$\vec{\nabla}\Psi=\vec{e}_r\dfrac{\partial\Psi}{\partial r}+\dfrac{1}{r}\vec{e}_\theta\dfrac{\partial\Psi}{\partial\theta}+\vec{e}_z\dfrac{\partial\Psi}{\partial z}$
Gradient of a vector, $\vec{A}$	$\vec{\nabla}\vec{A}=\begin{bmatrix}\dfrac{\partial A(r)}{\partial r} & \dfrac{\partial A(\theta)}{\partial r} & \dfrac{\partial A(z)}{\partial r}\\ \dfrac{\partial A(r)}{r\partial\theta}-\dfrac{A(\theta)}{r} & \dfrac{\partial A(\theta)}{r\partial\theta}+\dfrac{A(r)}{r} & \dfrac{\partial A(z)}{r\partial\theta}\\ \dfrac{\partial A(r)}{\partial z} & \dfrac{\partial A(\theta)}{\partial z} & \dfrac{\partial A(z)}{\partial z}\end{bmatrix}$

(continued)

Curl of a vector, $\vec{A}$	$\vec{\nabla}\times\vec{A}=\left(\frac{\partial A(z)}{r\partial\theta}-\frac{\partial A(\theta)}{\partial z}\right)\vec{e}(r)+\left(\frac{\partial A(r)}{\partial z}-\frac{\partial A(z)}{\partial r}\right)\vec{e}(\theta)+\left(\frac{\partial(rA(\theta))}{r\partial r}-\frac{\partial A(r)}{r\partial\theta}\right)\vec{e}(z)$
Divergence of a vector, $\vec{A}$	$\vec{\nabla}\cdot\vec{A}=\frac{\partial(rA(r))}{r\partial r}+\frac{\partial A(\theta)}{r\partial\theta}+\frac{\partial A(z)}{\partial z}$
Laplacian operator	$\nabla^2=\vec{\nabla}\cdot(\vec{\nabla})=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$
Laplacian of a scalar, Ψ	$\nabla^2\Psi=\vec{\nabla}\cdot(\vec{\nabla}\Psi)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}+\frac{\partial^2\Psi}{\partial z^2}$
Laplacian of a vector, $\vec{A}$	$\nabla^2\vec{A}=\left[\nabla^2A(r)-\frac{2}{r^2}\frac{\partial A(\theta)}{\partial\theta}-\frac{A(r)}{r^2}\right]\vec{e}(r)+\left[\nabla^2A(\theta)+\frac{2}{r^2}\frac{\partial A(r)}{\partial\theta}-\frac{A(\theta)}{r^2}\right]\vec{e}(\theta)+\left[\nabla^2A(z)\right]\vec{e}(z)$
Biharmonic operator	$\nabla^4=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]\right]+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left[\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]+\frac{\partial^4}{\partial z^4}$

14.3 SPHERICAL COORDINATE SYSTEMS

We use $(\vec{e}(1),\vec{e}(2),\vec{e}(3))=(\vec{e}_r,\vec{e}_\varphi,\vec{e}_\theta)$, $(A(1),A(2),A(3))=(A(r),A(\varphi),A(\theta))$ to designate the unit vectors $\vec{e}(i)$ and physical components of vector $\vec{A}$. Most of the expressions are written in terms of physical components. For Christoffel symbols, see Equation 10.28.

TABLE 14.3 Relations for quantities and their related derivatives in spherical coordinate systems.

Coordinates (orthogonal)	(r,φ,θ), θ azimuth and φ polar angle		
Coordinate functions	$x=r\sin\varphi\cos\theta$	$y=r\sin\varphi\sin\theta$	$z=r\cos\varphi$
Coordinate surfaces	$x^2+y^2+z^2=r^2$, spheres	$(x^2+y^2)/z^2=\tan^2\varphi$, cones	$y/x=\tan\theta$, planes

(continued)

Basis vectors, $\vec{e}_i$, covariant	$\left(\vec{e}_r,\vec{e}_\varphi,\vec{e}_\theta\right)=\left(\begin{Bmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{Bmatrix},\begin{Bmatrix}r\cos\varphi\cos\theta\\ r\cos\varphi\sin\theta\\ -r\sin\varphi\end{Bmatrix},\begin{Bmatrix}-r\sin\varphi\sin\theta\\ r\sin\varphi\cos\theta\\ 0\end{Bmatrix}\right)$
Basis vectors, $\vec{e}^i$, contravariant	$\left(\vec{e}^r,\vec{e}^\varphi,\vec{e}^\theta\right)=\left(\begin{Bmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{Bmatrix},\frac{1}{r}\begin{Bmatrix}\cos\varphi\cos\theta\\ \cos\varphi\sin\theta\\ -\sin\varphi\end{Bmatrix},\frac{1}{r\sin\varphi}\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix}\right)$
Scale factors, h_i	$\left(h_r,h_\varphi,h_\theta\right)=\left(1,r,r\sin\varphi\right)$
Unit vectors, physical components	$\left(\vec{e}(r),\vec{e}(\varphi),\vec{e}(\theta)\right)=\left(\begin{Bmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{Bmatrix},\begin{Bmatrix}\cos\varphi\cos\theta\\ \cos\varphi\sin\theta\\ -\sin\varphi\end{Bmatrix},\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix}\right)$
Metric tensors, g_{ij} and g^{ij}	$g_{rr}=\left(g^{rr}\right)^{-1}=1, g_{\varphi\varphi}=\left(g^{\varphi\varphi}\right)^{-1}=r^2, g_{\theta\theta}=\left(g^{\theta\theta}\right)^{-1}=\left(r\sin\varphi\right)^2$, the rest are null-system is orthogonal
Jacobian	$\mathcal{J}=r^2\sin\varphi$
Unit volume	$d\mathbb{V}=r^2\sin\varphi\, dr d\varphi d\theta$

Line element and magnitude	$\overrightarrow{dS}=dr\vec{e}(r)+rd\varphi\vec{e}(\varphi)+r\sin\varphi d\theta\vec{e}(\theta)$	$\left\lvert\overrightarrow{dS}\right\rvert=\sqrt{(dr)^2+(rd\varphi)^2+(r\sin\varphi d\theta)^2}$

Vector components	$\vec{A}=A(r)\vec{e}(r)+A(\varphi)\vec{e}(\varphi)+A(\theta)\vec{e}(\theta)$

Christoffel symbols, based on covariant Basis vectors	$\Gamma^r_{\varphi\varphi}=-r$	$\Gamma^r_{\theta\theta}=-r\sin^2\varphi$	$\Gamma^\varphi_{\theta\theta}=-\sin\varphi\cos\varphi$	$\Gamma^\varphi_{r\varphi}=\Gamma^\theta_{r\theta}=1/r$	$\Gamma^\theta_{\varphi\theta}=\cot\varphi$
Christoffel symbols, based on Unit vectors	$\Gamma^r_{\varphi\varphi}=-1$	$\Gamma^r_{\theta\theta}=-\sin\varphi$	$\Gamma^\varphi_{\theta\theta}=-\cos\varphi$	$\Gamma^\varphi_{r\varphi}=1$, $\Gamma^\theta_{r\theta}=\sin\varphi$	$\Gamma^\theta_{\varphi\theta}=\cos\varphi$

Dot-product (two vectors)	$\vec{A}\cdot\vec{B}=A(r)B(r)+A(\varphi)B(\varphi)+A(\theta)B(\theta)$
Cross-product (two vectors)	$\vec{A}\times\vec{B}=\left[A(\varphi)B(\theta)-A(\theta)B(\varphi)\right]\vec{e}(r)+\left[A(\theta)B(r)-A(r)B(\theta)\right]\vec{e}(\varphi)+\left[A(r)B(\varphi)-A(\varphi)B(r)\right]\vec{e}(\theta)$

(continued)

Gradient vector	$\vec{\nabla}=\left(\frac{\partial}{\partial r}\right)\vec{e}_r+\left(\frac{1}{r}\frac{\partial}{\partial\varphi}\right)\vec{e}_\varphi+\left(\frac{1}{r\sin\varphi}\frac{\partial}{\partial\theta}\right)\vec{e}_\theta$
Gradient of a scalar, Ψ	$\vec{\nabla}\Psi=\frac{\partial\Psi}{\partial r}\vec{e}_r+\left(\frac{1}{r}\frac{\partial\Psi}{\partial\varphi}\right)\vec{e}_\varphi+\left(\frac{1}{r\sin\varphi}\frac{\partial\Psi}{\partial\theta}\right)\vec{e}_\theta$
Gradient of a vector, $\vec{A}$	$\vec{\nabla}\vec{A}=$ $\begin{bmatrix}\frac{\partial A(r)}{\partial r} & \frac{\partial A(\varphi)}{\partial r} & \frac{\partial A(\theta)}{\partial r}\\ \frac{\partial A(r)}{r\partial\varphi}-\frac{A(\varphi)}{r} & \frac{\partial A(\varphi)}{r\partial\varphi}+\frac{A(r)}{r} & \frac{\partial A(\theta)}{r\partial\varphi}\\ \frac{1}{r\sin\varphi}\frac{\partial A(r)}{\partial\theta}-\frac{A(\theta)}{r} & \frac{1}{r\sin\varphi}\frac{\partial A(\varphi)}{\partial\theta}-\frac{A(\theta)}{r}\cot\varphi & \frac{1}{r\sin\varphi}\frac{\partial A(\theta)}{\partial\theta}+\frac{A(r)}{r}+\frac{A(\varphi)}{r}\cot\varphi\end{bmatrix}$
Curl	$\vec{\nabla}\times\vec{A}=\frac{1}{r\sin\varphi}\left(\frac{\partial\left(A(\theta)\sin\varphi\right)}{\partial\varphi}-\frac{\partial A(\varphi)}{\partial\theta}\right)\vec{e}(r)$ $+\frac{1}{r}\left(\frac{1}{\sin\varphi}\frac{\partial A(r)}{\partial\theta}-\frac{\partial\left(rA(\theta)\right)}{\partial r}\right)\vec{e}(\varphi)$ $+\frac{1}{r}\left(\frac{\partial\left(rA(\varphi)\right)}{\partial r}-\frac{\partial A(r)}{\partial\varphi}\right)\vec{e}(\theta)$
Divergence of a vector, $\vec{A}$	$\vec{\nabla}\cdot\vec{A}=\frac{1}{r^2}\frac{\partial\left(r^2A(r)\right)}{\partial r}+\frac{1}{r\sin\varphi}\frac{\partial\left(A(\varphi)\sin\varphi\right)}{\partial\varphi}+\frac{1}{r\sin\varphi}\frac{\partial A(\theta)}{\partial\theta}$
Divergence of a 2nd rank tensor, $\boldsymbol{A}$	$\vec{\nabla}\cdot\boldsymbol{A}=\left(\frac{\partial\left(r^2A(rr)\right)}{r^2\partial r}+\frac{1}{r\sin\varphi}\frac{\partial\left(A(r\varphi)\sin\varphi\right)}{r\partial\varphi}-\frac{A(\varphi\varphi)+A(\theta\theta)}{r}+\frac{1}{r\sin\varphi}\frac{\partial A(r\theta)}{\partial\theta}\right)\vec{e}(r)$ $+\left(\frac{\partial\left(r^2A(r\varphi)\right)}{r^2\partial r}+\frac{1}{r\sin\varphi}\frac{\partial\left(A(\varphi\varphi)\sin\varphi\right)}{r\partial\varphi}+\frac{A(r\varphi)}{r}+\frac{1}{r\sin\varphi}\frac{\partial A(\varphi\theta)}{\partial\theta}-\frac{A(\theta\theta)}{r}\cot\varphi\right)\vec{e}(\varphi)$ $+\left(\frac{\partial\left(r^2A(r\theta)\right)}{r^2\partial r}+\frac{\partial A(\varphi\theta)}{r\partial\varphi}+\frac{1}{r\sin\varphi}\frac{\partial A(\theta\theta)}{\partial\theta}+\frac{A(r\theta)}{r}+\frac{2A(\varphi\theta)}{r}\cot\varphi\right)\vec{e}(\theta)$
Laplacian operator	$\nabla^2=\vec{\nabla}\cdot\left(\vec{\nabla}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)+\frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}$
Laplacian of a scalar, Ψ	$\nabla^2\Psi=\vec{\nabla}\cdot\left(\vec{\nabla}\Psi\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\Psi}{\partial\varphi}\right)$ $+\frac{1}{r^2\sin^2\varphi}\frac{\partial^2\Psi}{\partial\theta^2}$

(continued)

Laplacian of a vector, $\vec{A}$	$\nabla^2\vec{A} = \left[\nabla^2 A(r) - \frac{2A(r)}{r^2} - \frac{2}{r^2\sin\varphi}\frac{\partial(\sin\varphi A(\varphi))}{\partial\varphi} - \frac{2}{r^2\sin\varphi}\frac{\partial A(\theta)}{\partial\theta}\right]\vec{e}(r)$ $+\left[\nabla^2 A(\varphi) - \frac{A(\varphi)}{r^2\sin^2\varphi} + + \frac{2}{r^2}\frac{\partial(A(r))}{\partial\varphi} - \frac{2\cos\varphi}{r^2\sin^2\varphi}\frac{\partial A(\theta)}{\partial\theta}\right]\vec{e}(\varphi)$ $+\left[\nabla^2 A(\theta) - \frac{A(\theta)}{r^2\sin^2\varphi} + \frac{2}{r^2\sin\varphi}\frac{\partial A(r)}{\partial\theta} + \frac{2\cos\varphi}{r^2\sin^2\varphi}\frac{\partial(A(\varphi))}{\partial\theta}\right]\vec{e}(\theta)$
Biharmonic operator, ∇^4	$\nabla^4 = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]\right]$ $+\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left[\sin\varphi\frac{\partial}{\partial\varphi}\left[\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)\right]\right]$ $+\frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}\left[\frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}\right]$

14.4 PARABOLIC COORDINATE SYSTEMS

We use $(\vec{e}(1),\vec{e}(2),\vec{e}(3)) = (\vec{e}(\xi),\vec{e}(\eta),\vec{e}(\theta))$, $(A(1),A(2),A(3)) = (A(\xi),A(\eta),A(\theta))$ to designate the unit vectors $\vec{e}(i)$ and physical components of vector $\vec{A}$. Most of the expressions are written in terms of physical components. For Christoffel symbols, see Example 15.4.

TABLE 14.4 Relations for quantities and their related derivatives in parabolic coordinate systems.

Coordinates (orthogonal)	(ξ,η,θ)		
Coordinate functions	$x = \xi\eta\cos\theta$	$y = \xi\eta\sin\theta$	$z = (\xi^2 - \eta^2)/2$
Coordinate surfaces	$x^2 + y^2 = -2\xi^2(z - \xi^2/2)$, paraboloids	$x^2 + y^2 = -2\eta^2(z - \eta^2/2)$, paraboloids	$y/x = \tan\theta$, planes
Basis vectors, $\vec{e}_i$-covariant	$(\vec{e}_\xi,\vec{e}_\eta,\vec{e}_\theta) = \left(\begin{Bmatrix}\eta\cos\theta\\ \eta\sin\theta\\ \xi\end{Bmatrix},\begin{Bmatrix}\xi\cos\theta\\ \xi\sin\theta\\ -\eta\end{Bmatrix},\begin{Bmatrix}-\xi\eta\sin\theta\\ \xi\eta\cos\theta\\ 0\end{Bmatrix}\right)$		

(continued)

Basis vectors, $\vec{e}^{\,i}$, contravariant	$\left(\vec{e}^{\,\xi},\vec{e}^{\,\eta},\vec{e}^{\,\theta}\right)=\left(\left(\dfrac{1}{\xi^2+\eta^2}\begin{Bmatrix}\eta\cos\theta\\ \eta\sin\theta\\ \xi\end{Bmatrix},\dfrac{1}{\xi^2+\eta^2}\begin{Bmatrix}\xi\cos\theta\\ \xi\sin\theta\\ -\eta\end{Bmatrix},\dfrac{1}{\xi\eta}\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix}\right)\right)$
Scale factors, h_i	$\left(h_\xi,h_\eta,h_\theta\right)=\left(\sqrt{\xi^2+\eta^2},\sqrt{\xi^2+\eta^2},\xi\eta\right)$
Unit vectors, physical components	$\left(\vec{e}(\xi),\vec{e}(\eta),\vec{e}(\theta)\right)=$ $\left(\dfrac{1}{\sqrt{\xi^2+\eta^2}}\begin{Bmatrix}\eta\cos\theta\\ \eta\sin\theta\\ \xi\end{Bmatrix},\dfrac{1}{\sqrt{\xi^2+\eta^2}}\begin{Bmatrix}\xi\cos\theta\\ \xi\sin\theta\\ -\eta\end{Bmatrix},\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix}\right)$
Metric tensors, g_{ij} and g^{ij}	$g_{\xi\xi}=\left(g^{\xi\xi}\right)^{-1}=\xi^2+\eta^2, g_{\eta\eta}=\left(g^{\eta\eta}\right)^{-1}=\xi^2+\eta^2, g_{\theta\theta}=\left(g^{\theta\theta}\right)^{-1}=(\xi\eta)^2$, the rest are null
Basis vectors, $\vec{e}^{\,i}$, contravariant	$\left(\vec{e}^{\,\xi},\vec{e}^{\,\eta},\vec{e}^{\,\theta}\right)=\left(\left(\dfrac{1}{\xi^2+\eta^2}\begin{Bmatrix}\eta\cos\theta\\ \eta\sin\theta\\ \xi\end{Bmatrix},\dfrac{1}{\xi^2+\eta^2}\begin{Bmatrix}\xi\cos\theta\\ \xi\sin\theta\\ -\eta\end{Bmatrix},\dfrac{1}{\xi\eta}\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix}\right)\right)$
Jacobian	$\mathcal{J}=\xi\eta\left(\xi^2+\eta^2\right)$
Unit volume	$d\mathbb{V}=\xi\eta\left(\xi^2+\eta^2\right)d\xi\,d\eta\,d\theta$
Line element and magnitude	$\overrightarrow{dS}=\sqrt{\xi^2+\eta^2}\left[d\xi\,\vec{e}(\xi)+d\eta\,\vec{e}(\eta)\right]+\xi\eta\,d\theta\,\vec{e}(\theta)$ $\left\lvert\overrightarrow{dS}\right\rvert=\sqrt{\left(\xi^2+\eta^2\right)(d\xi)^2+\left(\xi^2+\eta^2\right)(d\eta)^2+(\xi\eta\,d\theta)^2}$
Vector components	$\vec{A}=A(\xi)\vec{e}(\xi)+A(\eta)\vec{e}(\eta)+A(\theta)\vec{e}(\theta)$
Christoffel symbols, based on covariant Basis vectors	$\Gamma^{\xi}_{\xi\xi}=-\Gamma^{\xi}_{\eta\eta}=\Gamma^{\eta}_{\xi\eta}=\Gamma^{\eta}_{\eta\xi}=\xi/\left(\xi^2+\eta^2\right)$ $\Gamma^{\xi}_{\xi\eta}=\Gamma^{\xi}_{\eta\xi}=\Gamma^{\eta}_{\eta\eta}=\eta/\left(\xi^2+\eta^2\right)$ $\Gamma^{\xi}_{\theta\theta}=-\xi\eta^2/\left(\xi^2+\eta^2\right)$ $\Gamma^{\eta}_{\theta\theta}=-\eta\xi^2/\left(\xi^2+\eta^2\right)$ $\Gamma^{\theta}_{\xi\theta}=\Gamma^{\theta}_{\theta\xi}=1/\xi$ $\Gamma^{\theta}_{\eta\theta}=\Gamma^{\theta}_{\theta\eta}=1/\eta$ $\Gamma^{\theta}_{\eta\theta}=\Gamma^{\theta}_{\theta\eta}=1/\eta$
Dot-product (two vectors)	$\vec{A}\cdot\vec{B}=A(\xi)B(\xi)+A(\eta)B(\eta)+A(\theta)B(\theta)$

(continued)

Cross-product (two vectors)	$\vec{A}\times\vec{B}=\left[A(\eta)B(\theta)-A(\theta)B(\eta)\right]\vec{e}(\xi)$ $+\left[A(\theta)B(\xi)-A(\xi)B(\theta)\right]\vec{e}(\eta)$ $+\left[A(\xi)B(\eta)-A(\eta)B(\xi)\right]\vec{e}(\theta)$
Gradient vector	$\vec{\nabla}=\left(\frac{1}{\sqrt{\xi^2+\eta^2}}\frac{\partial}{\partial r}\right)\vec{e}_\xi+\left(\frac{1}{\sqrt{\xi^2+\eta^2}}\frac{\partial}{\partial\eta}\right)\vec{e}_\eta+\left(\frac{1}{\xi\eta}\frac{\partial}{\partial\varphi}\right)\vec{e}_\varphi$
Gradient of a scalar, Ψ	$\vec{\nabla}\Psi=\left(\frac{1}{\sqrt{\xi^2+\eta^2}}\frac{\partial\Psi}{\partial r}\right)\vec{e}_\xi+\left(\frac{1}{\sqrt{\xi^2+\eta^2}}\frac{\partial\Psi}{\partial\eta}\right)\vec{e}_\eta+\left(\frac{1}{\xi\eta}\frac{\partial\Psi}{\partial\varphi}\right)\vec{e}_\varphi$
Laplacian operator	$\nabla^2=\vec{\nabla}\cdot\left(\vec{\nabla}\right)=\frac{1}{\xi\left(\xi^2+\eta^2\right)}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right)+\frac{1}{\eta\left(\xi^2+\eta^2\right)}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right)$ $+\frac{1}{(\xi\eta)^2}\frac{\partial^2}{\partial\theta^2}$
Laplacian of a scalar, Ψ	$\nabla^2\Psi=\vec{\nabla}\cdot\left(\vec{\nabla}\Psi\right)=\frac{1}{\xi\left(\xi^2+\eta^2\right)}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Psi}{\partial\xi}\right)+\frac{1}{\eta\left(\xi^2+\eta^2\right)}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\Psi}{\partial\eta}\right)$ $+\frac{1}{(\xi\eta)^2}\frac{\partial^2\Psi}{\partial\theta^2}$
Laplacian of a vector, $\vec{A}$	$\nabla^2\vec{A}=$, use Equation 11.21.

For this coordinate system, we didn't list all relations, since they are lengthy. But readers can use the relations given in the next section, as they are applicable to all orthogonal coordinate systems.

14.5 ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS

We use $(x_1,x_2,x_3)\equiv\left(x^1,x^2,x^3\right)$ symbols for presentation. Note that all components listed in Table 14.5 are given as physical components. The functional relations for coordinate variables vs. those of Cartesian ones are required for calculating the values of quantities listed in Table 14.5.

TABLE 14.5 Relations for quantities and their related derivatives in curvilinear orthogonal coordinate systems.

Coordinates (orthogonal)	$(x_1, x_2, x_3) \equiv (x^1, x^2, x^3)$	
Basis vectors $\vec{e}_i$, covariant	$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	
Scale factors, h_i	(h_1, h_2, h_3)	
Unit vectors $\vec{e}(i)$, physical components	$(\vec{e}(1), \vec{e}(2), \vec{e}(3)) = (\vec{e}_1 / h_1, \vec{e}_2 / h_2, \vec{e}_3 / h_3)$	
Metric tensors, g_{ij} and g^{ij}	$g_{11} = (g^{11})^{-1} = (h_1)^2$, $g_{22} = (g^{22})^{-1} = (h_2)^2$, $g_{33} = (g^{33})^{-1} = (h_3)^2$, the rest are null	
Basis vectors $\vec{e}^i$ contravariant	$(\vec{e}^1, \vec{e}^2, \vec{e}^3) = (\vec{e}_1 g^{11}, \vec{e}_2 g^{22}, \vec{e}_3 g^{33})$	
Jacobian	$\mathcal{J} = h_1 h_2 h_3$	
Unit volume	$d\mathbb{V} = \mathcal{J} dx_1 dx_2 dx_3$	
Line element and magnitude	$\overrightarrow{dS} = h_1 dx_1 \vec{e}(1) + h_2 dx_2 \vec{e}(2) + h_3 dx_3 \vec{e}(3)$	$\lvert\overrightarrow{dS}\rvert = \sqrt{(h_1 dx_1)^2 + (h_2 dx_2)^2 + (h_3 dx_3)^2}$
Vector components	$\vec{A} = A(1)\vec{e}(1) + A(2)\vec{e}(2) + A(3)\vec{e}(3)$	
Christoffel symbols	(use Equations 10.20-10.25)	
Dot-product (two vectors)	$\vec{A} \cdot \vec{B} = A(1)B(1) + A(2)B(2) + A(3)B(3)$	
Cross-product (two vectors)	$\vec{A} \times \vec{B} = [A(2)B(3) - A(3)B(2)]\vec{e}(1) + [A(3)B(1) - A(1)B(3)]\vec{e}(2) + [A(1)B(2) - A(2)B(1)]\vec{e}(3)$	
Gradient vector	$\vec{\nabla} = \frac{1}{h_1}\frac{\partial}{\partial x_1}\vec{e}(1) + \frac{1}{h_2}\frac{\partial}{\partial x_2}\vec{e}(2) + \frac{1}{h_3}\frac{\partial}{\partial x_3}\vec{e}(3)$	
Gradient of a scalar, Ψ	$\vec{\nabla}\Psi = \frac{1}{h_1}\frac{\partial \Psi}{\partial x_1}\vec{e}(1) + \frac{1}{h_2}\frac{\partial \Psi}{\partial x_2}\vec{e}(2) + \frac{1}{h_3}\frac{\partial \Psi}{\partial x_3}\vec{e}(3)$	

(continued)

Gradient of a vector, $\vec{A}$	$\vec{\nabla}\vec{A}=\left(\nabla_j\vec{A}\right)\vec{e}^{\,j}=\nabla_j\left(A^i\vec{e}_i\right)\vec{e}^{\,j}=\left(\dfrac{\partial A^i}{\partial x^j}+A^k\Gamma^i_{kj}\right)\vec{e}_i\vec{e}^{\,j}$, see Example 15.13.
Curl of a vector, $\vec{A}$	$\vec{\nabla}\times\vec{A}=\dfrac{1}{h_2h_3}\left[\dfrac{\partial\left(h_3A(3)\right)}{\partial x_2}-\dfrac{\partial\left(h_2A(2)\right)}{\partial x_3}\right]\vec{e}(1)+\dfrac{1}{h_1h_3}\left[\dfrac{\partial\left(h_1A(1)\right)}{\partial x_3}-\dfrac{\partial\left(h_3A(3)\right)}{\partial x_1}\right]\vec{e}(2)+\dfrac{1}{h_1h_2}\left[\dfrac{\partial\left(h_2A(2)\right)}{\partial x_1}-\dfrac{\partial\left(h_1A(1)\right)}{\partial x_2}\right]\vec{e}(3)$
Divergence of a vector, $\vec{A}$	$\vec{\nabla}\cdot\vec{A}=\dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial\left(h_2h_3A(1)\right)}{\partial x_1}+\dfrac{\partial\left(h_1h_3A(2)\right)}{\partial x_2}+\dfrac{\partial\left(h_1h_2A(3)\right)}{\partial x_3}\right]$
Divergence of a 2nd rank tensor, $\boldsymbol{A}$	$\vec{\nabla}\cdot\boldsymbol{A}=\left[\dfrac{1}{h_1h_2h_3}\left(\dfrac{\partial\left(h_2h_3A(11)\right)}{\partial x_1}+\dfrac{\partial\left(h_1h_3A(21)\right)}{\partial x_2}+\dfrac{\partial\left(h_1h_2A(31)\right)}{\partial x_3}\right)+\dfrac{A(12)}{h_1h_2}\dfrac{\partial h_1}{\partial x_2}+\dfrac{A(31)}{h_1h_3}\dfrac{\partial h_1}{\partial x_3}-\dfrac{A(22)}{h_1h_2}\dfrac{\partial h_2}{\partial x_1}-\dfrac{A(33)}{h_1h_3}\dfrac{\partial h_3}{\partial x_1}\right]\vec{e}(1)+\left[\dfrac{1}{h_1h_2h_3}\left(\dfrac{\partial\left(h_2h_3A(12)\right)}{\partial x_1}+\dfrac{\partial\left(h_1h_3A(22)\right)}{\partial x_2}+\dfrac{\partial\left(h_1h_2A(32)\right)}{\partial x_3}\right)+\dfrac{A(23)}{h_2h_3}\dfrac{\partial h_2}{\partial x_3}+\dfrac{A(12)}{h_1h_2}\dfrac{\partial h_2}{\partial x_1}-\dfrac{A(33)}{h_2h_3}\dfrac{\partial h_3}{\partial x_2}-\dfrac{A(11)}{h_1h_2}\dfrac{\partial h_1}{\partial x_2}\right]\vec{e}(2)+\left[\dfrac{1}{h_1h_2h_3}\left(\dfrac{\partial\left(h_2h_3A(13)\right)}{\partial x_1}+\dfrac{\partial\left(h_1h_3A(23)\right)}{\partial x_2}+\dfrac{\partial\left(h_1h_2A(33)\right)}{\partial x_3}\right)+\dfrac{A(31)}{h_1h_3}\dfrac{\partial h_3}{\partial x_1}+\dfrac{A(23)}{h_2h_3}\dfrac{\partial h_3}{\partial x_2}-\dfrac{A(11)}{h_3h_1}\dfrac{\partial h_1}{\partial x_3}-\dfrac{A(22)}{h_2h_3}\dfrac{\partial h_2}{\partial x_3}\right]\vec{e}(3)$
Laplacian of a scalar, Ψ	$\nabla^2\Psi=\vec{\nabla}\cdot\left(\vec{\nabla}\Psi\right)=\dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial x_1}\left(\dfrac{h_2h_3}{h_1}\dfrac{\partial\Psi}{\partial x_1}\right)+\dfrac{\partial}{\partial x_2}\left(\dfrac{h_1h_3}{h_2}\dfrac{\partial\Psi}{\partial x_2}\right)+\dfrac{\partial}{\partial x_3}\left(\dfrac{h_1h_2}{h_3}\dfrac{\partial\Psi}{\partial x_3}\right)\right]$

(continued)

Laplacian of a vector, $\vec{A}$	$\nabla^2\vec{A} = \left\{\frac{1}{h_1}\frac{\partial}{\partial x_1}\left(\vec{\nabla}\cdot\vec{A}\right)+\frac{1}{h_2h_3}\left[\begin{matrix}\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left(\frac{\partial(h_1A_1)}{\partial x_3}-\frac{\partial(h_3A_3)}{\partial x_1}\right)\right)\\-\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left(\frac{\partial(h_2A_2)}{\partial x_1}-\frac{\partial(h_1A_1)}{\partial x_2}\right)\right)\end{matrix}\right]\right\}\vec{e}_1 + \left\{\frac{1}{h_2}\frac{\partial}{\partial x_2}\left(\vec{\nabla}\cdot\vec{A}\right)+\frac{1}{h_1h_3}\left[\begin{matrix}\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1h_2}\left(\frac{\partial(h_2A_2)}{\partial x_1}-\frac{\partial(h_1A_1)}{\partial x_2}\right)\right)\\-\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2h_3}\left(\frac{\partial(h_3A_3)}{\partial x_2}-\frac{\partial(h_2A_2)}{\partial x_3}\right)\right)\end{matrix}\right]\right\}\vec{e}_2 + \left\{\frac{1}{h_3}\frac{\partial}{\partial x_3}\left(\vec{\nabla}\cdot\vec{A}\right)+\frac{1}{h_2h_1}\left[\begin{matrix}\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left(\frac{\partial(h_3A_3)}{\partial x_2}-\frac{\partial(h_2A_2)}{\partial x_3}\right)\right)\\-\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left(\frac{\partial(h_1A_1)}{\partial x_3}-\frac{\partial(h_3A_3)}{\partial x_1}\right)\right)\end{matrix}\right]\right\}\vec{e}_3$
Biharmonic operator, ∇^4	$\nabla^4 = \frac{1}{\mathcal{J}}\frac{\partial}{\partial x^k}\left\{\mathcal{J}g^{kl}\frac{\partial}{\partial x^l}\underbrace{\left[\frac{1}{\mathcal{J}}\frac{\partial}{\partial x^i}\left(\mathcal{J}g^{ij}\frac{\partial}{\partial x^j}\right)\right]}_{\nabla^2}\right\} = \frac{1}{h_1h_2h_3}\left\{\begin{matrix}\frac{\partial}{\partial x_1}\left[\frac{h_2h_3}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial x_1}\right)\right]\right]\\+\frac{\partial}{\partial x_2}\left[\frac{h_1h_3}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1h_2h_3}\frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial x_2}\right)\right]\right]\\+\frac{\partial}{\partial x_3}\left[\frac{h_1h_2}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial x_3}\right)\right]\right]\end{matrix}\right\}$

Modern engineering software tools provide facilities to include curvilinear coordinate systems for computer modeling. For example, COMSOL Multiphysics® has a *curvilinear coordinate interface* for defining object orientation in an arbitrary system [12].

CHAPTER 15

WORKED-OUT EXAMPLES

In this section, we present several worked-out examples related to the topics covered in the previous sections.

15.1 EXAMPLE: EINSTEIN SUMMATION CONVENTION

Write out the expanded expression in full detail for $N=3$ dimensions for, $A = dx^i\left(\vec{e}_i - \frac{\partial x'^j}{\partial x^i}\vec{e'}_j\right)$.

Solution:

Summation applies on both indices i and j, hence $A = dx^i\left(\vec{e}_i - \frac{\partial x'^j}{\partial x^i}\vec{e'}_j\right) =$

$$dx^i\vec{e}_i - dx^i\frac{\partial x'^j}{\partial x^i}\vec{e'}_j = dx^1\vec{e}_1 + dx^2\vec{e}_2 + dx^3\vec{e}_3 - dx^1\left(\frac{\partial x'^1}{\partial x^1}\vec{e'_1} + \frac{\partial x'^2}{\partial x^1}\vec{e'_2} + \frac{\partial x'^3}{\partial x^1}\vec{e'_3}\right) -$$

$$dx^2\left(\frac{\partial x'^1}{\partial x^2}\vec{e'_1} + \frac{\partial x'^2}{\partial x^2}\vec{e'_2} + \frac{\partial x'^3}{\partial x^2}\vec{e'_3}\right) - dx^3\left(\frac{\partial x'^1}{\partial x^3}\vec{e'_1} + \frac{\partial x'^2}{\partial x^3}\vec{e'_2} + \frac{\partial x'^3}{\partial x^3}\vec{e'_3}\right).$$

Re-arranging the terms, we get $A = dx^1\left(\vec{e}_1 - \frac{\partial x'^1}{\partial x^1}\vec{e'_1} - \frac{\partial x'^2}{\partial x^1}\vec{e'_2} - \frac{\partial x'^3}{\partial x^1}\vec{e'_3}\right) +$

$$dx^2\left(\vec{e}_2 - \frac{\partial x'^1}{\partial x^2}\vec{e'_1} - \frac{\partial x'^2}{\partial x^2}\vec{e'_2} - \frac{\partial x'^3}{\partial x^2}\vec{e'_3}\right) + dx^3\left(\vec{e}_3 - \frac{\partial x'^1}{\partial x^3}\vec{e'_1} - \frac{\partial x'^2}{\partial x^3}\vec{e'_2} - \frac{\partial x'^3}{\partial x^3}\vec{e'_3}\right).$$

15.2 EXAMPLE: CONVERSION FROM VECTOR TO INDEX NOTATIONS

Write out the following expressions written in vector form in index notation and expand for 3D coordinates.

a. $\left(\vec{\nabla}\cdot\vec{A}\right)^2$

b. $\vec{A}\cdot\left(\nabla^2\vec{A}\right)$

c. $\vec{A}\cdot\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\right)$

d. $\left(\vec{\nabla}\cdot\vec{A}\right)\left(\nabla^2\Phi\right)$

e. $\nabla^2\left(\vec{A}\cdot(\nabla\Phi)\right)$

f. $\vec{A}\cdot\left(\vec{\nabla}\left(\nabla^2\Phi\right)\right)$

Solution:

a. $\left(\vec{\nabla}\cdot\vec{A}\right)^2 = A^j_{,j}A^k_{,k} = \left(A^1_{,1}+A^2_{,2}+A^3_{,3}\right)^2$

b. $$\vec{A}\cdot\left(\nabla^2\vec{A}\right) = A^kA_{k,\cdot j}^{\quad j} = A^1A_{1,\cdot j}^{\quad j} + A^2A_{2,\cdot j}^{\quad j} + A^3A_{3,\cdot j}^{\quad j} = A^1\left(A_{1,\cdot 1}^{\quad 1} + A_{1,\cdot 2}^{\quad 2} + A_{1,\cdot 3}^{\quad 3}\right)$$
$$+A^2\left(A_{1,\cdot 1}^{\quad 1} + A_{1,\cdot 2}^{\quad 2} + A_{1,\cdot 3}^{\quad 3}\right) + A^3\left(A_{1,\cdot 1}^{\quad 1} + A_{1,\cdot 2}^{\quad 2} + A_{1,\cdot 3}^{\quad 3}\right)$$

c. $$\vec{A}\cdot\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\right) = A^jA^k_{,kj} = A^1A^k_{,k1} + A^2A^k_{,k2} + A^3A^k_{,k3} = A^1\left(A^1_{,11}+A^2_{,21}+A^3_{,31}\right)$$
$$+A^2\left(A^1_{,11}+A^2_{,21}+A^3_{,31}\right) + A^3\left(A^1_{,11}+A^2_{,21}+A^3_{,31}\right)$$

d. $$\left(\vec{\nabla}\cdot\vec{A}\right)\left(\nabla^2\Phi\right) = A^j_{,j}\left(\nabla_k\Phi\right)_{,\cdot}^{\ k} = A^1_{,1}\left(\nabla_k\Phi\right)_{,\cdot}^{\ k} + A^2_{,2}\left(\nabla_k\Phi\right)_{,\cdot}^{\ k} + A^3_{,3}\left(\nabla_k\Phi\right)_{,\cdot}^{\ k} =$$
$$A^1_{,1}\left[\left(\nabla_1\Phi\right)_{,\cdot}^{\ 1} + \left(\nabla_2\Phi\right)_{,\cdot}^{\ 2} + \left(\nabla_3\Phi\right)_{,\cdot}^{\ 3}\right] + A^2_{,2}\left[\left(\nabla_1\Phi\right)_{,\cdot}^{\ 1} + \left(\nabla_2\Phi\right)_{,\cdot}^{\ 2} + \left(\nabla_3\Phi\right)_{,\cdot}^{\ 3}\right] +$$
$$A^3_{,3}\left[\left(\nabla_1\Phi\right)_{,\cdot}^{\ 1} + \left(\nabla_2\Phi\right)_{,\cdot}^{\ 2} + \left(\nabla_3\Phi\right)_{,\cdot}^{\ 3}\right]$$

e. $\nabla^2\left(\vec{A}\cdot(\nabla\Phi)\right)=\left(A^k(\nabla_k\Phi)\right)_{,j\cdot}^{\;\;j}$

$$=\left(A^1(\nabla_1\Phi)+A^2(\nabla_2\Phi)+A^3(\nabla_3\Phi)\right)_{,j\cdot}^{\;\;j}$$

$$=\left(A^1(\nabla_1\Phi)+A^2(\nabla_2\Phi)+A^3(\nabla_3\Phi)\right)_{,1\cdot}^{\;\;1}$$

$$+\left(A^1(\nabla_1\Phi)+A^2(\nabla_2\Phi)+A^3(\nabla_3\Phi)\right)_{,2\cdot}^{\;\;2}$$

$$+\left(A^1(\nabla_1\Phi)+A^2(\nabla_2\Phi)+A^3(\nabla_3\Phi)\right)_{,3\cdot}^{\;\;3}$$

f. $\vec{A}\cdot\left(\vec{\nabla}\left(\nabla^2\Phi\right)\right)=A^j(\nabla_k\Phi)_{,\cdot j}^{\;\;k}$

$$=A^1(\nabla_k\Phi)_{,\cdot 1}^{\;\;k}+A^2(\nabla_k\Phi)_{,\cdot 2}^{\;\;k}+A^3(\nabla_k\Phi)_{,\cdot 3}^{\;\;k}$$

$$=A^1\left[(\nabla_1\Phi)_{,\cdot 1}^{\;\;1}+(\nabla_2\Phi)_{,\cdot 1}^{\;\;2}+(\nabla_3\Phi)_{,\cdot 1}^{\;\;3}\right]$$

$$+A^2\left[(\nabla_1\Phi)_{,\cdot 2}^{\;\;1}+(\nabla_2\Phi)_{,\cdot 2}^{\;\;2}+(\nabla_3\Phi)_{,\cdot 2}^{\;\;3}\right]$$

$$+A^3\left[(\nabla_1\Phi)_{,\cdot 3}^{\;\;1}+(\nabla_2\Phi)_{,\cdot 3}^{\;\;2}+(\nabla_3\Phi)_{,\cdot 3}^{\;\;3}\right]$$

15.3 EXAMPLE: OBLIQUE RECTILINEAR COORDINATE SYSTEMS

Consider a 2D oblique/slanted coordinate system, x^i in which the coordinate axes are not at right angles. With reference to the Cartesian coordinate system y^i, as shown in Figure 15.1, for a given vector $\vec{A}$ find:

a. Covariant and contravariant components; A_i and A^i

b. Covariant and contravariant basis vectors; $\vec{e}_i$ and $\vec{e}^{\,i}$ with sketch on the coordinate system

c. Show that $\vec{A}=A^i\vec{e}_i=A_i\vec{e}^{\,i}=A_{c1}\vec{E}_1+A_{c2}\vec{E}_2$; where $\vec{E}_1$ and $\vec{E}_2$ are Cartesian unit vectors

d. Verify that $\vec{e}^{\,i}\cdot\vec{e}_j=\delta^{\,i}_{\,j}$ and sketch all basis vectors

e. Metric tensors

f. Unit vectors $\vec{e}_i/\left|\vec{e}_i\right|$, or scale factor, and $\vec{e}^{\,i}/\left|\vec{e}^{\,i}\right|$

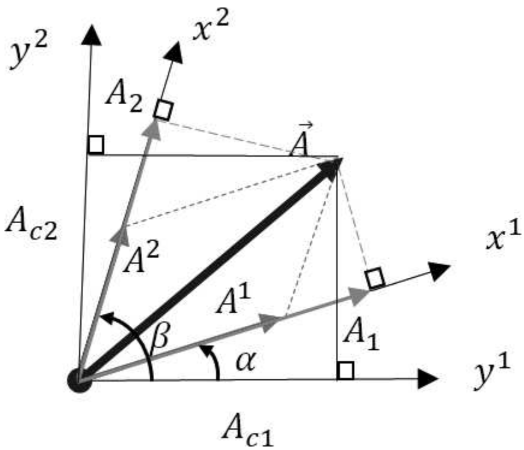


FIGURE 15.1 Oblique coordinate system x^i and Cartesian coordinate system y^i.

Solution:

Assume axes x^1 and x^2 make angles α and β with Cartesian axis y^1, respectively. Therefore, we can write the following functional relations between the oblique and Cartesian systems:

$$\begin{cases} y^1 = x^1 \cos\alpha + x^2 \cos\beta \\ y^2 = x^1 \sin\alpha + x^2 \sin\beta \end{cases}, \text{ or inversely } \begin{cases} x^1 = -\csc(\alpha-\beta)\left[y^1 \sin\beta - y^2 \cos\beta\right] \\ x^2 = \csc(\alpha-\beta)\left[y^1 \sin\alpha - y^2 \cos\alpha\right] \end{cases}$$

a. Covariant components of $\vec{A}$ are obtained by drawing perpendicular lines to the coordinate axes. We have $A_i = \frac{\partial y^j}{\partial x^i} A_{cj}$, or $A_1 = \frac{\partial y^1}{\partial x^1} A_{c1} + \frac{\partial y^2}{\partial x^1} A_{c2} = \cos\alpha\, A_{c1} + \sin\alpha\, A_{c2}$. Similarly, $A_2 = \frac{\partial y^1}{\partial x^2} A_{c1} + \frac{\partial y^2}{\partial x^2} A_{c2} = \cos\beta\, A_{c1} + \sin\beta\, A_{c2}$. The contravariant components of $\vec{A}$ are obtained by drawing parallel lines to the coordinate axes. We have $A^i = \frac{\partial x^i}{\partial y^j} A_{cj}$, or $A^1 = \frac{\partial x^1}{\partial y^1} A_{c1} + \frac{\partial x^1}{\partial y^2} A_{c2} = -\sin\beta \csc(\alpha-\beta) A_{c1} + \cos\beta \csc(\alpha-\beta) A_{c2}$. Similarly, $A^2 = \frac{\partial x^2}{\partial y^1} A_{c1} + \frac{\partial x^2}{\partial y^2} A_{c2} = \sin\alpha \csc(\alpha-\beta) A_{c1} - \cos\alpha \csc(\alpha-\beta) A_{c2}$. Note that for the Cartesian system both covariant and contravariant components are identical.

b. The covariant basis vectors for oblique system are $\vec{e}_i = \frac{\partial y^j}{\partial x^i}\vec{E}_j$, or $\vec{e}_1 = \frac{\partial y^1}{\partial x^1}\vec{E}_1 + \frac{\partial y^2}{\partial x^1}\vec{E}_2 = \cos\alpha\,\vec{E}_1 + \sin\alpha\,\vec{E}_2$. Similarly, $\vec{e}_2 = \frac{\partial y^1}{\partial x^2}\vec{E}_1 + \frac{\partial y^2}{\partial x^2}\vec{E}_2 = \cos\beta\,\vec{E}_1 + \sin\beta\,\vec{E}_2$. The contravariant basis vectors are $\vec{e}^i = \frac{\partial x^i}{\partial y^j}\vec{E}_j$, or $\vec{e}^1 = \frac{\partial x^1}{\partial y^1}\vec{E}_1 + \frac{\partial x^1}{\partial y^2}\vec{E}_2 = -\sin\beta\csc(\alpha-\beta)\vec{E}_1 + \cos\beta\csc(\alpha-\beta)\vec{E}_2$. Similarly, $\vec{e}^2 = \frac{\partial x^2}{\partial y^1}\vec{E}_1 + \frac{\partial x^2}{\partial y^2}\vec{E}_2 = \sin\alpha\csc(\alpha-\beta)\vec{E}_1 - \cos\alpha\csc(\alpha-\beta)\vec{E}_2$. Note that for the Cartesian system both covariant and contravariant basis vectors are identical and are unit vectors, i.e. $\left|\vec{E}_i\right| = 1$.

c. $A^i\vec{e}_i = A^1\vec{e}_1 + A^2\vec{e}_2 = [-\sin\beta\csc(\alpha-\beta)A_{c1} + \cos\beta\csc(\alpha-\beta)A_{c2}][\cos\alpha\,\vec{E}_1 + \sin\alpha\,\vec{E}_2] + [\sin\alpha\csc(\alpha-\beta)A_{c1} - \cos\alpha\csc(\alpha-\beta)A_{c2}][\cos\beta\,\vec{E}_1 + \sin\beta\,\vec{E}_2]$. After some manipulations and rearranging the terms, we find $A^i\vec{e}_i = A_{c1}\vec{E}_1 + A_{c2}\vec{E}_2$. Similarly, $A_i\vec{e}^i = A_1\vec{e}^1 + A_2\vec{e}^2 = [\cos\alpha\,A_{c1} + \sin\alpha\,A_{c2}][-\sin\beta\csc(\alpha-\beta)\vec{E}_1 + \cos\beta\csc(\alpha-\beta)\vec{E}_2] + [\cos\beta\,A_{c1} + \sin\beta\,A_{c2}][\sin\alpha\csc(\alpha-\beta)\vec{E}_1 - \cos\alpha\csc(\alpha-\beta)\vec{E}_2]$. After some manipulations and rearranging the terms, we find $A_i\vec{e}^i = A_{c1}\vec{E}_1 + A_{c2}\vec{E}_2$.

d. Dot-product of covariant and contravariant basis vectors can now be written as $\vec{e}_1\cdot\vec{e}^2 = \vec{e}^2\cdot\vec{e}_1 = \frac{\cos\alpha\sin\alpha}{\sin(\alpha-\beta)} - \frac{\sin\alpha\cos\alpha}{\sin(\alpha-\beta)} = 0$ and $\vec{e}_2\cdot\vec{e}^1 = \vec{e}^1\cdot\vec{e}_2 = -\frac{\cos\beta\sin\beta}{\sin(\alpha-\beta)} + \frac{\sin\beta\cos\beta}{\sin(\alpha-\beta)} = 0$. These results confirm that $\vec{e}_1 \perp \vec{e}^2$ and $\vec{e}_2 \perp \vec{e}^1$, as shown in Figure 15.2. Also, we can write $\vec{e}_1\cdot\vec{e}^1 = -\frac{\cos\alpha\sin\beta}{\sin(\alpha-\beta)} + \frac{\sin\alpha\cos\beta}{\sin(\alpha-\beta)} = 1$ and $\vec{e}_2\cdot\vec{e}^2 = \frac{\cos\beta\sin\alpha}{\sin(\alpha-\beta)} - \frac{\sin\beta\cos\alpha}{\sin(\alpha-\beta)} = 1$. These results verify that $\vec{e}^i\cdot\vec{e}_j = \delta^i_j$.

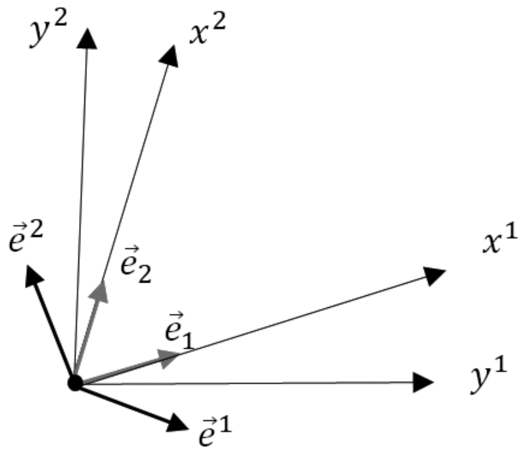


FIGURE 15.2 Covariant and contravariant basis vectors in oblique coordinate systems.

e. The doubly covariant metric tensor $g_{ij} = \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & \cos(\alpha - \beta) \\ \cos(\alpha - \beta) & 1 \end{bmatrix}$. We can calculate doubly contravariant metric tensor by inverting the g_{ij}, or using $g^{ij} = \begin{bmatrix} \vec{e}^1 \cdot \vec{e}^1 & \vec{e}^1 \cdot \vec{e}^2 \\ \vec{e}^2 \cdot \vec{e}^1 & \vec{e}^2 \cdot \vec{e}^2 \end{bmatrix} = \csc^2(\alpha - \beta) \begin{bmatrix} 1 & -\cos(\alpha - \beta) \\ -\cos(\alpha - \beta) & 1 \end{bmatrix}$. These results confirm that x^i is a non-orthogonal system, since off-diagonal elements of metric tensors are not equal to zero for all values of angles α and β with the condition that $(\alpha - \beta) \neq (2k+1)\frac{\pi}{2}$. where, k is an integer.

f. Unit vectors along the covariant basis vectors are; $\frac{\vec{e}_1}{|\vec{e}_1|} = \cos\alpha\, \vec{E}_1 + \sin\alpha\, \vec{E}_2$ and $\frac{\vec{e}_2}{|\vec{e}_2|} = \cos\beta\, \vec{E}_1 + \sin\beta\, \vec{E}_2$. The unit vectors along the contravariant basis vectors are; $\frac{\vec{e}^1}{|\vec{e}^1|} = \frac{\vec{e}^1}{\sqrt{\vec{e}^1 \cdot \vec{e}^1}} = -\sin\beta\, \vec{E}_1 + \cos\beta\, \vec{E}_2$ and $\frac{\vec{e}^2}{|\vec{e}^2|} = \frac{\vec{e}^2}{\sqrt{\vec{e}^2 \cdot \vec{e}^2}} = \sin\alpha\, \vec{E}_1 - \cos\alpha\, \vec{E}_2$.

15.4 EXAMPLE: QUANTITIES RELATED TO PARABOLIC COORDINATE SYSTEM

Consider a parabolic coordinate system $\left(x^1, x^2, x^3\right) \equiv (\xi, \eta, \theta)$ given in terms of Cartesian $\left(y^1, y^2, y^3\right) \equiv (X, Y, Z)$, as shown in Figure 15.3. The functional relations are

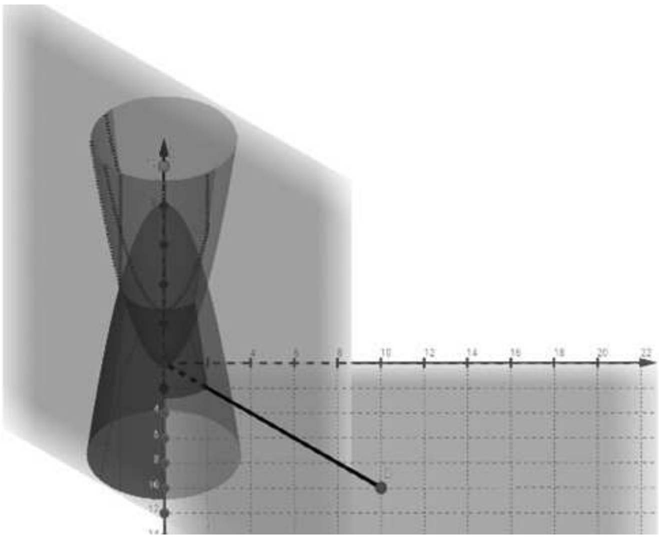


FIGURE 15.3 Parabolic coordinate system.

$$\begin{cases} X = \xi\eta \cos\theta \\ Y = \xi\eta \sin\theta \\ Z = \left(\xi^2 - \eta^2\right)/2 \end{cases}$$

. Find the basis vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ for the parabolic coordinate system in terms of the Cartesian unit vectors. Also find the scale factors, unit vectors, metric tensors (covariant and contravariant), Jacobian, volume element, and Christoffel symbols of the 2nd kind.

Solution:

The covariant basis vector $\left(\vec{e}_\xi, \vec{e}_\eta, \vec{e}_\theta\right)$ reads, using $\vec{e}_k = \dfrac{\partial y^j}{\partial x^k}\vec{E}_j$

$$\begin{cases} \vec{e}_\xi = \eta \cos\theta \vec{E}_1 + \eta \sin\theta \vec{E}_2 + \xi \vec{E}_3 \\ \vec{e}_\eta = \xi \cos\theta \vec{E}_1 + \xi \sin\theta \vec{E}_2 - \eta \vec{E}_3 \\ \vec{e}_\theta = -\xi\eta \sin\theta \vec{E}_1 + \xi\eta \cos\theta \vec{E}_2 \end{cases}$$

The scale factors are $h_\xi = \left|\vec{e}_\xi\right| = \sqrt{\xi^2 + \eta^2}$, $h_\eta = \left|\vec{e}_\eta\right| = \sqrt{\xi^2 + \eta^2}$, and $h_\theta = \xi\eta$. The unit vectors are, using $\vec{e}(k) = \vec{e}_k / h_k$,

$$\begin{cases} \vec{e}(\xi) = \left(\eta \cos\theta \vec{E}_1 + \eta \sin\theta \vec{E}_2 + \xi \vec{E}_3\right) / \sqrt{\xi^2 + \eta^2} \\ \vec{e}(\eta) = \left(\xi \cos\theta \vec{E}_1 + \xi \sin\theta \vec{E}_2 - \eta \vec{E}_3\right) / \sqrt{\xi^2 + \eta^2} \\ \vec{e}(\theta) = -\sin\theta \vec{E}_1 + \cos\theta \vec{E}_2 \end{cases}$$

Covariant metric tensor reads, using $g_{ij} = \vec{e}_i \cdot \vec{e}_j$, $g_{ij} = \begin{bmatrix} \xi^2 + \eta^2 & 0 & 0 \\ 0 & \xi^2 + \eta^2 & 0 \\ 0 & 0 & (\xi\eta)^2 \end{bmatrix}$. Therefore, the parabolic coordinate system is an orthogonal one, since metric tensor is a diagonal matrix with null off-diagonal elements. The contravariant metric tensor is the inverse of the covariant one, or $g^{ij} = \left(g_{ij}\right)^{-1} = \begin{bmatrix} \left(\xi^2 + \eta^2\right)^{-1} & 0 & 0 \\ 0 & \left(\xi^2 + \eta^2\right)^{-1} & 0 \\ 0 & 0 & (\xi\eta)^{-2} \end{bmatrix}$.

Jacobian is the square root of the determinant of g_{ij}, or the product of all scale factors, $\mathcal{J} = \sqrt{\left|g_{ij}\right|} = \xi\eta\left(\xi^2 + \eta^2\right)$. The volume element is $d\mathbb{V} = \mathcal{J} d\xi d\eta d\theta = \xi\eta\left(\xi^2 + \eta^2\right) d\xi d\eta d\theta$. We use Equation 10.20, or $\Gamma_{ij}^m = \frac{g^{mk}}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right]$ to calculate the Christoffel symbols. Therefore, $\Gamma_{11}^1 = \frac{g^{11}}{2}\left[\frac{\partial g_{11}}{\partial \xi}\right] = \frac{\xi}{\xi^2 + \eta^2}$,

$\Gamma_{12}^1 = \frac{g^{11}}{2}\left[\frac{\partial g_{11}}{\partial \eta}\right] = \frac{\eta}{\xi^2 + \eta^2}$, $\Gamma_{13}^1 = 0$, $\Gamma_{11}^2 = \frac{g^{22}}{2}\left[-\frac{\partial g_{11}}{\partial \eta}\right] = -\frac{\eta}{\xi^2 + \eta^2}$,

$\Gamma_{12}^2 = \frac{g^{22}}{2}\left[\frac{\partial g_{22}}{\partial \xi}\right] = \frac{\xi}{\xi^2 + \eta^2}$, $\Gamma_{13}^2 = \Gamma_{11}^3 = \Gamma_{12}^3 = 0$, $\Gamma_{13}^3 = \frac{g^{33}}{2}\left[\frac{\partial g_{33}}{\partial \xi}\right] = \frac{1}{\xi}$,

$$\Gamma^1_{22} = \frac{g^{11}}{2}\left[-\frac{\partial g_{22}}{\partial \xi}\right] = -\frac{\xi}{\xi^2+\eta^2}, \qquad \Gamma^1_{23} = 0, \qquad \Gamma^1_{33} = \frac{g^{11}}{2}\left[-\frac{\partial g_{33}}{\partial \xi}\right] = -\frac{\xi\eta^2}{\xi^2+\eta^2},$$

$$\Gamma^2_{22} = \frac{g^{22}}{2}\left[\frac{\partial g_{22}}{\partial \eta}\right] = \frac{\eta}{\xi^2+\eta^2}, \qquad \Gamma^2_{23} = 0, \qquad \Gamma^2_{33} = \frac{g^{22}}{2}\left[-\frac{\partial g_{33}}{\partial \eta}\right] = -\frac{\eta\xi^2}{\xi^2+\eta^2},$$

$\Gamma^3_{23} = \frac{g^{33}}{2}\left[\frac{\partial g_{33}}{\partial \eta}\right] = \frac{1}{\eta}$, $\Gamma^3_{33} = 0$. In matrix form, we have

$$\Gamma^\xi = \begin{bmatrix} \Gamma^\xi_{\xi\xi} & \Gamma^\xi_{\xi\eta} & \Gamma^\xi_{\xi\theta} \\ \Gamma^\xi_{\eta\xi} & \Gamma^\xi_{\eta\eta} & \Gamma^\xi_{\eta\theta} \\ \Gamma^\xi_{\theta\xi} & \Gamma^\xi_{\theta\eta} & \Gamma^\xi_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \frac{\xi}{\xi^2+\eta^2} & \frac{\eta}{\xi^2+\eta^2} & 0 \\ \frac{\eta}{\xi^2+\eta^2} & -\frac{\xi}{\xi^2+\eta^2} & 0 \\ 0 & 0 & -\frac{\xi\eta^2}{\xi^2+\eta^2} \end{bmatrix}$$

$$\Gamma^\eta = \begin{bmatrix} \Gamma^\eta_{\xi\xi} & \Gamma^\eta_{\xi\eta} & \Gamma^\eta_{\xi\theta} \\ \Gamma^\eta_{\eta\xi} & \Gamma^\eta_{\eta\eta} & \Gamma^\eta_{\eta\theta} \\ \Gamma^\eta_{\theta\xi} & \Gamma^\eta_{\theta\eta} & \Gamma^\eta_{\theta\theta} \end{bmatrix} = \begin{bmatrix} -\frac{\eta}{\xi^2+\eta^2} & \frac{\xi}{\xi^2+\eta^2} & 0 \\ \frac{\xi}{\xi^2+\eta^2} & \frac{\eta}{\xi^2+\eta^2} & 0 \\ 0 & 0 & -\frac{\eta\xi^2}{\xi^2+\eta^2} \end{bmatrix}$$

$$\Gamma^\theta = \begin{bmatrix} \Gamma^\theta_{\xi\xi} & \Gamma^\theta_{\xi\eta} & \Gamma^\theta_{\xi\theta} \\ \Gamma^\theta_{\eta\xi} & \Gamma^\theta_{\eta\eta} & \Gamma^\theta_{\eta\theta} \\ \Gamma^\theta_{\theta\xi} & \Gamma^\theta_{\theta\eta} & \Gamma^\theta_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{\xi} \\ 0 & 0 & \frac{1}{\eta} \\ \frac{1}{\xi} & \frac{1}{\eta} & 0 \end{bmatrix}$$

15.5 EXAMPLE: QUANTITIES RELATED TO BI-POLAR COORDINATE SYSTEMS

Consider a bi-polar coordinate system $\left(x^1,x^2,x^3\right)\equiv(\xi,\eta,z)$ given in terms of Cartesian $\left(y^1,y^2,y^3\right)\equiv(X,Y,Z)$, as shown in Figure 15.4. The functional relations are $\begin{cases} X=\dfrac{a\sinh\eta}{\cosh\eta-\cos\xi} \\ Y=\dfrac{a\sin\xi}{\cosh\eta-\cos\xi} \\ Z=z \end{cases}$. Find the basis vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ for the bi-polar coordinate system in terms of the Cartesian unit vectors. Also find the scale factors, unit vectors, metric tensors (covariant and contravariant), Jacobian, volume element, and Christoffel symbols of the 2nd kind.

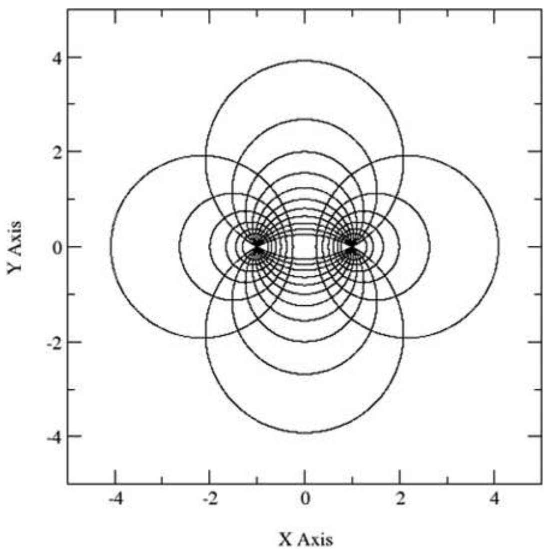


FIGURE 15.4 Bipolar coordinates: ξ and η iso-surfaces (Foci are located at (−1, 0) and (1, 0)[1].

[1] (https://commons.wikimedia.org/wiki/File:Bipolar_isosurfaces.png)

Solution:

The covariant basis vector $\left(\vec{e}_\xi, \vec{e}_\eta, \vec{e}_z\right)$ reads, using $\vec{e}_k = \dfrac{\partial y^j}{\partial x^k}\vec{E}_j$

$$\begin{cases} \vec{e}_\xi = \dfrac{-a\sinh\eta\,\sin\xi}{(\cosh\eta - \cos\xi)^2}\vec{E}_1 + \dfrac{a(\cosh\eta\,\cos\xi - 1)}{(\cosh\eta - \cos\xi)^2}\vec{E}_2 \\ \vec{e}_\eta = \dfrac{a(1-\cosh\eta\,\cos\xi)}{(\cosh\eta - \cos\xi)^2}\vec{E}_1 - \dfrac{a\sinh\eta\,\sin\xi}{(\cosh\eta - \cos\xi)^2}\vec{E}_2 \\ \vec{e}_z = \vec{E}_3 \end{cases}$$

The scale factors, after simplification, are $h_\xi = \left|\vec{e}_\xi\right| = \dfrac{a}{\cosh\eta - \cos\xi}$, $h_\eta = \left|\vec{e}_\eta\right| = \dfrac{a}{\cosh\eta - \cos\xi}$, and $h_z = 1$. The unit vectors are, using $\vec{e}(k) = \vec{e}_k / h_k$,

$$\begin{cases} \vec{e}(\xi) = \dfrac{-\sinh\eta\,\sin\xi}{\cosh\eta - \cos\xi}\vec{E}_1 + \dfrac{\cosh\eta\,\cos\xi - 1}{\cosh\eta - \cos\xi}\vec{E}_2 \\ \vec{e}(\eta) = \dfrac{1-\cosh\eta\,\cos\xi}{\cosh\eta - \cos\xi}\vec{E}_1 - \dfrac{\sinh\eta\,\sin\xi}{\cosh\eta - \cos\xi}\vec{E}_2 \\ \vec{e}(z) = \vec{E}_3 \end{cases}$$

The covariant metric tensor reads, using $g_{ij} = \vec{e}_i \cdot \vec{e}_j$,

$$g_{ij} = \begin{bmatrix} \dfrac{a^2}{(\cosh\eta - \cos\xi)^2} & 0 & 0 \\ 0 & \dfrac{a^2}{(\cosh\eta - \cos\xi)^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, the parabolic coordinate system is an orthogonal one, since metric tensor is a diagonal matrix with null off-diagonal elements. The contravariant metric tensor is the inverse of the covariant one, or $g^{ij} = \left(g_{ij}\right)^{-1} = \begin{bmatrix} \dfrac{\left(\cosh\eta - \cos\xi\right)^2}{a^2} & 0 & 0 \\ 0 & \dfrac{\left(\cosh\eta - \cos\xi\right)^2}{a^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Jacobian is the square root of the determinant of g_{ij}, or the product of all scale factors, $\mathcal{J} = \sqrt{\left|g_{ij}\right|} = \dfrac{a^2}{\left(\cosh\eta - \cos\xi\right)^2}$. The volume element is $d\mathbb{V} = \mathcal{J} d\xi\, d\eta\, d\theta = \dfrac{a^2}{\left(\cosh\eta - \cos\xi\right)^2} d\xi\, d\eta\, dz$. Similarly, as given in Example 15.5, the Christoffel symbols can be calculated as

$$\Gamma^{\xi} = \begin{bmatrix} \Gamma^{\xi}_{\xi\xi} & \Gamma^{\xi}_{\xi\eta} & \Gamma^{\xi}_{\xi z} \\ \Gamma^{\xi}_{\eta\xi} & \Gamma^{\xi}_{\eta\eta} & \Gamma^{\xi}_{\eta z} \\ \Gamma^{\xi}_{z\xi} & \Gamma^{\xi}_{z\eta} & \Gamma^{\xi}_{zz} \end{bmatrix} = \begin{bmatrix} \dfrac{\sin\xi}{\cos\xi - \cosh\eta} & \dfrac{\sinh\eta}{\cos\xi - \cosh\eta} & 0 \\ \dfrac{\sinh\eta}{\cos\xi - \cosh\eta} & \dfrac{-\sin\xi}{\cos\xi - \cosh\eta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{\eta} = \begin{bmatrix} \Gamma^{\eta}_{\xi\xi} & \Gamma^{\eta}_{\xi\eta} & \Gamma^{\eta}_{\xi z} \\ \Gamma^{\eta}_{\eta\xi} & \Gamma^{\eta}_{\eta\eta} & \Gamma^{\eta}_{\eta z} \\ \Gamma^{\eta}_{z\xi} & \Gamma^{\eta}_{z\eta} & \Gamma^{\eta}_{zz} \end{bmatrix} = \begin{bmatrix} \dfrac{-\sinh\eta}{\cos\xi - \cosh\eta} & \dfrac{\sin\xi}{\cos\xi - \cosh\eta} & 0 \\ \dfrac{\sin\xi}{\cos\xi - \cosh\eta} & \dfrac{\sinh\eta}{\cos\xi - \cosh\eta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{z} = [0]$$

15.6 EXAMPLE: APPLICATION OF CONTRAVARIANT METRIC TENSORS

Using the property of contravariant metric tensors, show that the following relation holds, $g^{il}\varepsilon_{ljk}A^jB^k = \varepsilon^{ijk}A_jB_k$.

Solution:

Knowing that metric tensors replace repeated indices with the free index while raising (contravariant metric tensor) or lowering (covariant metric tensor) them. We can write the expression in the L.H.S of the above relation as, $\underbrace{g^{il}\varepsilon_{ljk}}A^jB^k = \mathcal{E}^i_{jk}A^jB^k$. But we can write $A^j = g^{jn}A_n$ and $B^k = g^{km}B_m$, substituting, gives $g^{il}\varepsilon_{ljk}A^jB^k =$ $\mathcal{E}^i_{jk}g^{jn}A_ng^{km}B_m = \underbrace{\mathcal{E}^i_{jk}g^{jn}}g^{km}A_nB_m = \underbrace{\mathcal{E}^{in}_{k}g^{km}}A_nB_m = \mathcal{E}^{inm}A_nB_m$, after changing dummy indices, i.e. $n \to j$ and $m \to k$, we get $\mathcal{E}^{inm}A_nB_m = \mathcal{E}^{ijk}A_jB_k$. Therefore, the given relation holds.

15.7 EXAMPLE: DOT AND CROSS PRODUCTS IN CYLINDRICAL AND SPHERICAL COORDINATES

Given that $A(i)$ and $B(i)$ are the physical components of vectors $\vec{A}$ and $\vec{B}$, respectively, write down the expressions for physical components of the cross-product, $\vec{A}\times\vec{B}$, and dot-product $\vec{A}\cdot\vec{B}$ in terms of $A(i)$ and $B(i)$, for the following coordinate systems (assume a 3D space, $N = 3$):

a. Cylindrical polar $\left(x^1, x^2, x^3\right) \equiv (r, \theta, z)$

b. Spherical polar $\left(x^1, x^2, x^3\right) \equiv (r, \varphi, \theta)$

Solution:

The dot-product and cross-product relations are $\vec{A}\cdot\vec{B} = A^iB_i$, $C^i = \mathcal{E}^{ijk}A_jB_k$ (contravariant component). We use the relations between covariant/contravariant and physical components, or $A_i = h_iA(i)$ and $A^i = A(i)/h_i$.

a. For cylindrical coordinates, we have the scale factors as $(h_1, h_2, h_3) = (1, r, 1)$ and the Jacobian is $\mathcal{J} = r$ (see Section 11.6.1). We can write $B_1 = h_1B(1) = B(1)$, $B_2 = h_2B(2) = rA(2)$, $B_3 = h_3B(3) = B(3)$.

Similarly, $A^1 = \frac{A(1)}{h_1} = A(1)$, $A^2 = \frac{A(2)}{h_2} = \frac{A(2)}{r}$, $A^3 = \frac{A(3)}{h_3} = A(3)$. Therefore, the dot-product reads

$\vec{A} \cdot \vec{B} = A^i B_i = A^1 B_1 + A^2 B_2 + A^3 B_3 = A(1)B(1) + A(2)B(2) + A(3)B(3)$, a scalar quantity. The physical component of the cross-product is $C(i) = h_i C^i = h_i \mathcal{E}^{ijk} A_j B_k$, no-summation on index i. Expanding the expression gives $C(1) = h_1 C^1 = h_1 \mathcal{E}^{1jk} A_j B_k = \frac{e_{1jk}}{\mathcal{J}} A_j B_k = \frac{1}{r}\left[h_2 A(2) h_3 B(3) - h_3 A(3) h_2 B(2)\right] = A(2)B(3) - A(3)B(2)$, and $C(2) = h_2 C^2 = r\mathcal{E}^{2jk} A_j B_k = r\frac{e_{2jk}}{\mathcal{J}} A_j B_k = \left[h_3 A(3) h_1 B(1) - h_1 A(1) h_3 B(3)\right] = A(3)B(1) - A(1)B(3)$, and $C(3) = h_3 C^3 = h_3 \mathcal{E}^{3jk} A_j B_k = \frac{e_{3jk}}{\mathcal{J}} A_j B_k = \frac{1}{r}\left[h_1 A(1) h_2 B(2) - h_2 A(2) h_1 B(1)\right] = A(1)B(2) - A(2)B(1)$.

b. For spherical coordinates, we have the scale factors as $(h_1, h_2, h_3) = (1, r, r\sin\varphi)$ and the Jacobian is $\mathcal{J} = r^2 \sin\varphi$, see section 11.6.1. We can write $B_1 = h_1 B(1) = B(1)$, $B_2 = h_2 B(2) = rA(2)$, $B_3 = h_3 B(3) = B(3)$. Similarly, $A^1 = \frac{A(1)}{h_1} = A(1)$, $A^2 = \frac{A(2)}{h_2} = \frac{A(2)}{r}$, $A^3 = \frac{A(3)}{h_3} = A(3)$. Therefore, the dot-product reads

$\vec{A} \cdot \vec{B} = A^i B_i = A^1 B_1 + A^2 B_2 + A^3 B_3 = A(1)B(1) + A(2)B(2) + A(3)B(3)$, a scalar quantity.

The physical component of the cross-product is $C(i) = h_i C^i = h_i \mathcal{E}^{ijk} A_j B_k$, no-summation on index i. Expanding the expression gives $C(1) = h_1 C^1 = h_1 \mathcal{E}^{1jk} A_j B_k = \frac{e_{1jk}}{\mathcal{J}} A_j B_k = \frac{1}{r^2 \sin\varphi}\left[h_2 A(2) h_3 B(3) - h_3 A(3) h_2 B(2)\right] =$

$A(2)B(3)-A(3)B(2)$, and $C(2)=h_2C^2=r\mathcal{E}^{2jk}A_jB_k=r\dfrac{e_{2jk}}{\mathcal{J}}A_jB_k=$ $\left[h_3A(3)h_1B(1)-h_1A(1)h_3B(3)\right]=A(3)B(1)-A(1)B(3)$, and $C(3)=$ $h_3C^3=h_3\mathcal{E}^{3jk}A_jB_k=r\sin\varphi\dfrac{e_{3jk}}{\mathcal{J}}A_jB_k=\dfrac{1}{r}\left[h_1A(1)h_2B(2)-h_2A(2)h_1B(1)\right]=$ $A(1)B(2)-A(2)B(1)$.

15.8 EXAMPLE: RELATION BETWEEN JACOBIAN AND METRIC TENSOR DETERMINANTS

Show that determinant g of metric tensor g_{ij} is equal to the square of Jacobian $\mathcal{J}$, in a coordinate system, when measured against Cartesian system.

Solution:

Let's consider two arbitrary system x^i and x'^i. We can write the metric tensor transformation as $g'_{ij}=\dfrac{\partial x^k}{\partial x'^i}\dfrac{\partial x^l}{\partial x'^j}g_{kl}$, since it is a doubly covariant tensor. Taking determinant of the both sides gives $\left|g'_{ij}\right|=\left|\dfrac{\partial x^k}{\partial x'^i}\right|\left|\dfrac{\partial x^l}{\partial x'^j}\right|\left|g_{kl}\right|$. But $\left|\dfrac{\partial x^k}{\partial x'^i}\right|$ or $\left|\dfrac{\partial x^l}{\partial x'^j}\right|$ is the Jacobian of transformation, hence $\left|g'_{ij}\right|=\mathcal{J}^2\left|g_{kl}\right|$. Rewriting the expression and using $\left|g_{kl}\right|=g$ and $\left|g'_{ij}\right|=g'$, we get $g'=\mathcal{J}^2g$, or $\mathcal{J}=\sqrt{\dfrac{g'}{g}}$.

Now if the original x^i system is the Cartesian system, then $g_{cartesian}=1$ and we get $g'=\mathcal{J}^2$.

15.9 EXAMPLE: DETERMINANT OF METRIC TENSORS USING DISPLACEMENT VECTORS

For an arbitrary coordinate system $x^i \equiv (\xi, \eta, \zeta)$ the magnitude of displacement vector is given by $ds^2 = 5d\xi^2 + 3d\eta^2 + 7d\zeta^2 - 4d\xi\, d\eta - 8d\xi\, d\zeta + 2d\eta\, d\zeta$. Find g and g^{ij} (i.e., the determinant of covariant metric tensor and the contravariant metric tensor).

Solution:

We have $ds^2 = g_{ij}dx^i dx^j$, expanding for the given coordinate system and using $g_{ij} = g_{ji}$, gives $ds^2 = g_{11}d\xi^2 + g_{22}d\eta^2 + g_{33}d\zeta^2 + 2g_{12}d\xi\, d\eta + 2g_{13}d\xi\, d\zeta + 2g_{23}d\eta\, d\zeta$. Therefore, after comparison we get $g_{11} = 5$, $g_{22} = 3$, $g_{33} = 7$, $g_{12} = -2$, $g_{13} = -4$, and $g_{23} = 1$. The matrix form of covariant metric tensor is $g_{ij} = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 3 & 1 \\ -4 & 1 & 7 \end{bmatrix}$. Determinant of $g_{ij} = g = 5(21-1) + 2(-14+4) - 4(-2+12) = 40$. Note that the given system is not orthogonal, since off-diagonal elements of g_{ij} are not zero. For contravariant metric tensor, we find the inverse of $g_{ij} = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 3 & 1 \\ -4 & 1 & 7 \end{bmatrix}$, or $g^{ij} = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 3 & 1 \\ -4 & 1 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{19}{40} & \frac{3}{40} \\ \frac{1}{4} & \frac{3}{40} & \frac{11}{40} \end{bmatrix}$.

This is calculated using the formula $g^{ij} = \dfrac{C_{ij}}{g} = \dfrac{\textit{cofactor of } g_{ij}}{\textit{determinante of } g_{ij}}$, for example; $g^{11} = \dfrac{C_{11}}{40} = \dfrac{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}{40} = \dfrac{21-1}{40} = \dfrac{1}{2}$, $g^{23} = \dfrac{C_{23}}{40} = \dfrac{-\begin{vmatrix} 5 & -2 \\ -4 & 1 \end{vmatrix}}{40} = \dfrac{-5+8}{40} = \dfrac{3}{40}$, etc.

15.10 EXAMPLE: DETERMINANT OF A 4 × 4 MATRIX USING PERMUTATION SYMBOLS

Calculate determinant of matrix $M = \begin{bmatrix} 5 & 8 & 15 & 21 \\ 4 & 10 & 13 & 11 \\ 3 & 6 & 9 & 12 \\ 2 & 4 & 7 & 1 \end{bmatrix}$, using Equation 9.13.

Solution:

We expand based on row, hence $|M| = e_{ijkl} M_{1i} M_{2j} M_{3k} M_{4l}$. Expanding we get $|M| = e_{1234} M_{11} M_{22} M_{33} M_{44} + e_{1243} M_{11} M_{22} M_{34} M_{43} + e_{1342} M_{11} M_{23} M_{34} M_{42} + e_{1324} M_{11} M_{23} M_{32} M_{44} + e_{1423} M_{11} M_{24} M_{32} M_{43} + e_{1432} M_{11} M_{24} M_{33} M_{42} + e_{2134} M_{12} M_{21} M_{33} M_{44} + e_{2143} M_{12} M_{21} M_{34} M_{43} + e_{2341} M_{12} M_{23} M_{34} M_{41} + e_{2314} M_{12} M_{23} M_{31} M_{44} + e_{2413} M_{12} M_{24} M_{31} M_{43} + e_{2431} M_{12} M_{24} M_{33} M_{41} + e_{3124} M_{13} M_{21} M_{32} M_{44} + e_{3142} M_{13} M_{21} M_{34} M_{42} + e_{3241} M_{13} M_{22} M_{34} M_{41} + e_{3214} M_{13} M_{22} M_{31} M_{44} + e_{3412} M_{13} M_{24} M_{31} M_{42} + e_{3421} M_{13} M_{24} M_{32} M_{41} + e_{4123} M_{14} M_{21} M_{32} M_{43} + e_{4132} M_{14} M_{21} M_{33} M_{42} + e_{4231} M_{14} M_{22} M_{33} M_{41} + e_{4213} M_{14} M_{22} M_{31} M_{43} + e_{4312} M_{14} M_{23} M_{31} M_{42} + e_{4321} M_{14} M_{23} M_{32} M_{41}$. Substituting for the values of the matrix elements and the permutation symbols (i.e., $+1$ for even and -1 for odd number of interchanges for indices), we get $|M| = 18$. Note that the number of terms out of expansion is equal to $4! = 4 \times 3 \times 2 = 24$, or in general $N!$ for an $N \times N$ matrix.

15.11 EXAMPLE: TIME DERIVATIVES OF THE JACOBIAN

Show that the time derivative of the Jacobian reads $\frac{1}{\mathcal{J}} \frac{\partial \mathcal{J}}{\partial t} = \vec{\nabla} \cdot \vec{V}$, where $\vec{V}$ is the velocity vector.

Solution:

Considering the transformation from coordinate system x^i to that of x'^i, we can write the Jacobian as the determinant of the transformation, using Equation 4.4, or $\mathcal{J} = \varepsilon^{knl} \frac{\partial x^1}{\partial x'^k} \frac{\partial x^2}{\partial x'^n} \frac{\partial x^3}{\partial x'^l}$. Taking the time derivative, we have

$$\frac{\partial \mathcal{J}}{\partial t} = \varepsilon^{knl} \left[\frac{\partial}{\partial x'^k} \left(\frac{\partial x^1}{\partial t} \right) \frac{\partial x^2}{\partial x'^n} \frac{\partial x^3}{\partial x'^l} + \frac{\partial x^1}{\partial x'^k} \frac{\partial}{\partial x'^n} \left(\frac{\partial x^2}{\partial t} \right) \frac{\partial x^3}{\partial x'^l} + \frac{\partial x^1}{\partial x'^k} \frac{\partial x^2}{\partial x'^n} \frac{\partial}{\partial x'^l} \left(\frac{\partial x^3}{\partial t} \right) \right].$$

But we can write $\dfrac{\partial x^1}{\partial t} = V^1, \dfrac{\partial x^2}{\partial t} = V^2, \dfrac{\partial x^3}{\partial t} = V^3$, i.e., contravariant components of velocity vector. Hence, using chain-rule or $\dfrac{\partial x^i}{\partial x'^n} = \dfrac{\partial x^i}{\partial x^j}\dfrac{\partial x^j}{\partial x'^n}$ we can write $\dfrac{\partial \mathcal{J}}{\partial t} =$

$$\varepsilon^{knl}\left[\underbrace{\frac{\partial V^1}{\partial x^i}\frac{\partial x^i}{\partial x'^k}\frac{\partial x^2}{\partial x^j}\frac{\partial x^j}{\partial x'^n}\frac{\partial x^3}{\partial x^p}\frac{\partial x^p}{\partial x'^l}}_{\neq 0;\, i=1\ \&\, j=2\ \&\, p=3} + \underbrace{\frac{\partial x^1}{\partial x^j}\frac{\partial x^j}{\partial x'^k}\frac{\partial V^2}{\partial x^i}\frac{\partial x^i}{\partial x'^n}\frac{\partial x^3}{\partial x^p}\frac{\partial x^p}{\partial x'^l}}_{\neq 0;\, j=1\ \&\, i=2\ \&\, p=3} + \underbrace{\frac{\partial x^1}{\partial x^j}\frac{\partial x^j}{\partial x'^k}\frac{\partial x^2}{\partial x^i}\frac{\partial x^i}{\partial x'^n}\frac{\partial V^3}{\partial x^p}\frac{\partial x^p}{\partial x'^l}}_{\neq 0;\, j=1\ \&\, i=2\ \&\, p=3}\right].$$

Therefore, after rearranging the terms, in each group, we get

$$\frac{\partial \mathcal{J}}{\partial t} = \frac{\partial V^1}{\partial x^1}\left(\varepsilon^{knl}\frac{\partial x^1}{\partial x'^k}\frac{\partial x^2}{\partial x'^n}\frac{\partial x^3}{\partial x'^l}\right) + \frac{\partial V^2}{\partial x^2}\left(\varepsilon^{knl}\frac{\partial x^1}{\partial x'^k}\frac{\partial x^2}{\partial x'^n}\frac{\partial x^3}{\partial x'^l}\right) + \frac{\partial V^3}{\partial x^3}\left(\varepsilon^{knl}\frac{\partial x^1}{\partial x'^k}\frac{\partial x^2}{\partial x'^n}\frac{\partial x^3}{\partial x'^l}\right).$$

But the terms in each bracket is the Jacobian, and hence we get $\dfrac{\partial \mathcal{J}}{\partial t} = \mathcal{J}\left(\dfrac{\partial V^1}{\partial x^1} + \dfrac{\partial V^2}{\partial x^2} + \dfrac{\partial V^3}{\partial x^3}\right) = \mathcal{J}\underbrace{\left[\dfrac{1}{\mathcal{J}}\left(\mathcal{J}\dfrac{\partial V^i}{\partial x^i}\right)\right]}_{\vec{\nabla}\cdot\vec{V}} = \mathcal{J}\vec{\nabla}\cdot\vec{V}$, which yields

$\dfrac{1}{\mathcal{J}}\dfrac{\partial \mathcal{J}}{\partial t} = \vec{\nabla}\cdot\vec{V}$. For divergence of a vector see Equation 11.12.

15.12 EXAMPLE: COVARIANT DERIVATIVES OF A CONSTANT VECTOR

Show that for a constant vector $\vec{C}$, the covariant derivative $C^i_{,j}$ is not necessarily zero. Find the expression for its derivative.

Solution:

We have $\vec{C} = C^i\vec{e}_i$; writing its covariant derivative, we get $\dfrac{\partial}{\partial x^j}\left(C^i\vec{e}_i\right)\vec{e}^{\,j}$. The covariant component then reads as $\dfrac{\partial}{\partial x^j}\left(C^i\vec{e}_i\right) = \left(\dfrac{\partial C^i}{\partial x^j}\right)\vec{e}_i + C^i\dfrac{\partial}{\partial x^j}\left(\vec{e}_i\right) =$ $\left(\dfrac{\partial C^i}{\partial x^j}\right)\vec{e}_i + \underbrace{C^i\Gamma^k_{ij}\vec{e}_k}_{i\leftrightarrow k} = \vec{e}_i\underbrace{\left(\dfrac{\partial C^i}{\partial x^j} + C^k\Gamma^i_{kj}\right)}_{C^i_{,j}}$. Now, since $\vec{C}$ is a constant, the first term in the bracket vanishes but the second term does not. Therefore, we get $C^i_{,j} = C^k\Gamma^i_{kj}$.

15.13 EXAMPLE: COVARIANT DERIVATIVES OF PHYSICAL COMPONENTS OF A VECTOR

For the vector $\vec{A} = A^i\vec{e}_i = A(i)\vec{e}(i)$, write down the expression for $A(i)_{,j}$. Use the covariant derivative of the vector $\vec{A}$ in terms its physical components, or $\nabla_j\vec{A} = \nabla_j\left(A(i)\vec{e}(i)\right) = \vec{e}(i)A(i)_{,j}$.

Solution:

The covariant derivative, in terms of contravariant component of the vector is $\nabla_j\vec{A} = \nabla_j\left(A^i\vec{e}_i\right) = \left(\dfrac{\partial A^i}{\partial x^j} + A^k\Gamma^i_{kj}\right)\vec{e}_i$. Substituting for contravariant components $A^i = A(i)/h_i$ and covariant basis vector $\vec{e}_i = h_i\vec{e}(i)$, we get $\nabla_j\vec{A} =$

$$\left(\frac{\partial\left(\dfrac{A(i)}{h_i}\right)}{\partial x^j} + \frac{A(k)}{h_k}\Gamma^i_{kj}\right)h_i\vec{e}(i) = \left(\frac{\partial(A(i))}{\partial x^j} - \frac{A(i)}{h_i}\frac{\partial(h_i)}{\partial x^j} + \frac{h_i}{h_k}A(k)\Gamma^i_{kj}\right)\vec{e}(i).$$

Therefore, $A(i)_{,j} = \dfrac{\partial(A(i))}{\partial x^j} - \dfrac{A(i)}{h_i}\dfrac{\partial(h_i)}{\partial x^j} + \dfrac{h_i}{h_k}A(k)\Gamma^i_{kj} = \nabla_j A(i) - \dfrac{A(i)}{h_i}\nabla_j h_i + \dfrac{h_i}{h_k}A(k)\Gamma^i_{kj}$. Note that the Christoffel symbol could be written in terms of physical components/ Unit vectors, i.e., Γ^i_{kj} could be written in terms of $\vec{e}(i)$, designated by $\hat{\Gamma}^i_{kj}$ (see Example 10.4.1).

15.14 EXAMPLE: CONTINUITY EQUATIONS IN SEVERAL COORDINATE SYSTEMS

The continuity for a fluid with density ρ and velocity V^i is given as $\dfrac{\partial\rho}{\partial t} + \dfrac{1}{\mathcal{J}}\dfrac{\partial}{\partial x^i}\left(\mathcal{J}\rho V^i\right) = 0$. Write this equation in tensor notation for $N = 3$, and express the continuity equation in

a. Cartesian coordinates (x, y, z) with velocity physical components given as $\vec{V} = (V_x, V_y, V_z)$.

b. Cylindrical coordinates (r,θ,z) with velocity physical components given as $\vec{V}=(V_r,V_\theta,V_z)$.

c. Spherical coordinates (r,φ,θ) with velocity physical components given as $\vec{V}=(V_r,V_\varphi,V_\theta)$.

Solution:

Summation on i gives, $\dfrac{\partial\rho}{\partial t}+\dfrac{1}{\mathcal{J}}\left[\dfrac{\partial}{\partial x^1}\left(\mathcal{J}\rho V^1\right)+\dfrac{\partial}{\partial x^2}\left(\mathcal{J}\rho V^2\right)+\dfrac{\partial}{\partial x^3}\left(\mathcal{J}\rho V^3\right)\right]=0.$

a. Jacobian for a Cartesian system is unity and physical and contravariant components are identical, hence the continuity equation reads,

$$\frac{\partial\rho}{\partial t}+\left[\frac{\partial}{\partial x}(\rho V_x)+\frac{\partial}{\partial y}\left(\rho V_y\right)+\frac{\partial}{\partial z}(\rho V_z)\right]=0.$$

b. Jacobian for cylindrical systems is $\mathcal{J}=r$ and $V^1=V_r$, $V^2=V_\theta/r$, $V^3=V_z$. Hence, the continuity reads, after some manipulations,

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho V_\theta)+\frac{\partial}{\partial z}(\rho V_z)=0.$$

c. Jacobian for spherical systems is $\mathcal{J}=r^2\sin\varphi$ and $V^1=V_r$, $V^2=V_\varphi/r$, $V^3=V_\theta/(r\sin\varphi)$. Hence, the continuity reads, after some manipulations,

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2\sin\varphi}\left[\frac{\partial}{\partial r}\left(r^2\sin\varphi\rho V_r\right)+\frac{\partial}{\partial\varphi}\left(r\sin\varphi\rho V_\varphi\right)+\frac{\partial}{\partial\theta}(r\rho V_\theta)\right]=0.$$

15.15 EXAMPLE: 4D SPHERICAL COORDINATE SYSTEMS

A 4D spherical system $x^i\equiv(r,\psi,\theta,\varphi)$ with two polar angles ψ and θ range from 0 to π and azimuth angle φ from 0 to 2π, is related to the 4D Cartesian coordinates $y^i\equiv(x,y,z,w)$ as

$$\begin{cases} x=r\sin\psi\,\sin\theta\,\cos\varphi \\ y=r\sin\psi\,\sin\theta\,\sin\varphi \\ z=r\sin\psi\,\cos\theta \\ w=r\cos\psi \end{cases}$$

a. Find the Jacobian for the spherical system and volume of a 4D sphere in this coordinate system.

b. Find the expression for the Laplacian of a scalar in the spherical coordinates and write out it in detail.

Solution:

1. Jacobian reads $\mathcal{J}=\dfrac{\partial(x,y,z,w)}{\partial(r,\psi,\theta,\varphi)}=\begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \psi} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \varphi}\\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \psi} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \varphi}\\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \psi} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \varphi}\\ \dfrac{\partial w}{\partial r} & \dfrac{\partial w}{\partial \psi} & \dfrac{\partial w}{\partial \theta} & \dfrac{\partial w}{\partial \varphi}\end{vmatrix}=$

$$\begin{vmatrix}\sin\psi\,\sin\theta\,\cos\varphi & r\cos\psi\,\sin\theta\,\cos\varphi & r\sin\psi\,\sin\theta\,\cos\varphi & -r\sin\psi\,\sin\theta\,\sin\varphi\\ \sin\psi\,\sin\theta\,\sin\varphi & r\cos\psi\,\sin\theta\,\sin\varphi & r\sin\psi\,\cos\theta\,\sin\varphi & r\sin\psi\,\sin\theta\,\cos\varphi\\ \sin\psi\,\cos\theta & r\cos\psi\,\cos\theta & -r\sin\psi\,\sin\theta & 0\\ \cos\psi & -r\sin\psi & 0 & 0\end{vmatrix}=$$

$-r^3\sin^2\psi\,\sin\theta.$

The differential volume is $d\mathbb{V}=\mathcal{J}dr\,d\psi\,d\theta\,d\varphi$, with $0\le\theta\ \&\ \psi\le\pi$ and $0\le\varphi\le 2\pi$. Volume is then a quadruplet integral of $d\mathbb{V}$, or $\mathbb{V}=\int_0^R\left(-r^3\right)\int_0^\pi\sin^2\psi\,d\psi\int_0^\pi\sin\theta\,d\theta\int_0^{2\pi}d\varphi=\pi R^4\int_0^\pi\dfrac{1-\cos 2\varphi}{2}=\dfrac{\pi^2R^4}{2}.$

2. The expression for the Laplacian of scalar S reads $\nabla^2S=\dfrac{1}{\mathcal{J}}\nabla_i\left(\mathcal{J}g^{ij}\nabla_jS\right)$. This coordinate system is orthogonal, since $g_{ij}=0$ for $i\neq j$. For example $g_{12}=\vec{e}_1\cdot\vec{e}_2=r\sin\psi\,\cos\psi\,\sin^2\theta\,\cos^2\theta-r\sin\psi\,\cos\psi\,\sin^2\theta\,\sin^2\varphi+r\sin\psi\,\cos\psi$ $\cos^2\theta-r\sin\psi\,\cos\psi=0$. Therefore, $g_{ii}=\dfrac{1}{g^{ii}}=h_i^2$. For calculating $g_{ii}=\vec{e}_i\cdot\vec{e}_i$, need the covariant basis vectors, or

$$\begin{cases}\vec{e}_1=\sin\psi\,\sin\theta\,\cos\varphi\,\vec{E}_1+\sin\psi\,\sin\theta\,\sin\varphi\,\vec{E}_2+\sin\psi\,\cos\theta\,\vec{E}_3+\cos\psi\,\vec{E}_4\\ \vec{e}_2=r\cos\psi\,\sin\theta\,\cos\varphi\,\vec{E}_1+r\cos\psi\,\sin\theta\,\sin\varphi\,\vec{E}_2+r\cos\psi\,\cos\theta\,\vec{E}_3-r\sin\psi\,\vec{E}_4\\ \vec{e}_3=r\sin\psi\,\cos\theta\,\cos\varphi\,\vec{E}_1+r\sin\psi\,\cos\theta\,\sin\varphi\,\vec{E}_2-r\sin\psi\,\sin\theta\,\vec{E}_3\\ \vec{e}_4=-r\sin\psi\,\sin\theta\,\sin\varphi\,\vec{E}_1+r\sin\psi\,\sin\theta\,\cos\varphi\,\vec{E}_2\end{cases}$$

which yields, $g_{11}=1$, $g_{22}=r^2$, $g_{33}=r^2\sin^2\psi$, $g_{44}=r^2\sin^2\psi\sin^2\theta$. Now, $\nabla^2 S=\frac{1}{\mathcal{J}}\nabla_i\left(\mathcal{J}g^{ij}\nabla_j S\right)=\nabla^2 S=\frac{1}{\mathcal{J}}\nabla_i\left(\mathcal{J}g^{ii}\nabla_i S\right)=\frac{1}{\mathcal{J}}\nabla_1\left(\mathcal{J}g^{11}\nabla_1 S\right)+\frac{1}{\mathcal{J}}\nabla_2\left(\mathcal{J}g^{22}\nabla_2 S\right)+\frac{1}{\mathcal{J}}\nabla_3\left(\mathcal{J}g^{33}\nabla_3 S\right)+\frac{1}{\mathcal{J}}\nabla_4\left(\mathcal{J}g^{44}\nabla_4 S\right)$. Each term reads, after substitution

$$\frac{1}{\mathcal{J}}\nabla_1\left(\mathcal{J}g^{11}\nabla_1 S\right)=\frac{-1}{r^3\sin^2\psi\,\sin\theta}\frac{\partial}{\partial r}\left(-r^3\sin^2\psi\,\sin\theta\,\frac{\partial S}{\partial r}\right)=\frac{3}{r}\frac{\partial S}{\partial r}+\frac{\partial^2 S}{\partial r^2}$$

$$\frac{1}{\mathcal{J}}\nabla_2\left(\mathcal{J}g^{22}\nabla_2 S\right)=\frac{-1}{r^3\sin^2\psi\,\sin\theta}\frac{\partial}{\partial\psi}\left(-r^{-2}r^3\sin^2\psi\,\sin\theta\,\frac{\partial S}{\partial\psi}\right)$$
$$=\frac{1}{r^2}\left(\cot\psi\,\frac{\partial S}{\partial\psi}+\frac{\partial^2 S}{\partial\psi^2}\right)$$

$$\frac{1}{\mathcal{J}}\nabla_3\left(\mathcal{J}g^{33}\nabla_3 S\right)=\frac{-1}{r^3\sin^2\psi\,\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{-r^3\sin^2\psi\,\sin\theta}{r^2\sin^2\psi}\frac{\partial S}{\partial\theta}\right)$$
$$=\frac{1}{r^2\sin^2\psi}\left(\cot\theta\,\frac{\partial S}{\partial\theta}+\frac{\partial^2 S}{\partial\theta^2}\right)$$

$$\frac{1}{\mathcal{J}}\nabla_4\left(\mathcal{J}g^{44}\nabla_4 S\right)=\frac{-1}{r^3\sin^2\psi\,\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{-r^3\sin^2\psi\,\sin\theta}{r^2\sin^2\psi\,\sin^2\theta}\frac{\partial S}{\partial\varphi}\right)$$
$$=\frac{1}{r^2\sin^2\psi\,\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2}$$

Hence, $\nabla^2 S=\frac{3}{r}\frac{\partial S}{\partial r}+\frac{\partial^2 S}{\partial r^2}+\frac{1}{r^2}\left(\cot\psi\,\frac{\partial S}{\partial\psi}+\frac{\partial^2 S}{\partial\psi^2}\right)+\frac{1}{r^2\sin^2\psi}\left(\cot\theta\,\frac{\partial S}{\partial\theta}+\frac{\partial^2 S}{\partial\theta^2}\right)+\frac{1}{r^2\sin^2\psi\,\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2}$.

15.16 EXAMPLE: COMPLEX DOUBLE DOT-CROSS PRODUCT EXPRESSIONS

The expression $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times}\,\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times}$ is given. The notations mean that the two gradient vectors (marked by $\times$) are cross product to each other as well as the two vectors $\vec{A}$, then the resulted vectors of the cross-product operations are dotted. Write the expression in tensor notation and simplify the result.

Solution:

The two cross-product results are dotted; hence we require the contravariant component of the cross-product of the gradient's and the covariant component of the cross-product of the vectors $\vec{A}$. Therefore, we can write $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times}\,\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times} = \varepsilon^{ijk}\left(\varepsilon_{ilm}A^l A^m_{,k}\right)_{,j}$. Expanding the resulted expression yields, $\varepsilon^{ijk}\varepsilon_{ilm}\left(A^l A^m_{,k}\right)_{,j} = \delta^j_l\delta^k_m\left(A^l A^m_{,k}\right)_{,j} - \delta^j_m\delta^k_l\left(A^l A^m_{,k}\right)_{,j} = \left(A^l A^k_{,k}\right)_{,l} - \left(A^k A^j_{,k}\right)_{,j}$.

Performing the covariant derivative operations gives $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times}\,\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times} = A^l_{,l}A^k_{,k} + A^l A^k_{,kl} - A^k_{,j}A^j_{,k} - \underbrace{A^k A^j_{,kj}}_{k\to l,\; j\to k} = A^l_{,l}A^k_{,k} - A^k_{,j}A^j_{,k}$. The first term of the result can be written in vector notation as $A^l_{,l}A^k_{,k} = \left(\vec{\nabla}\cdot\vec{A}\right)^2$. Therefore, we finally have $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times}\,\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times} = \left(\vec{\nabla}\cdot\vec{A}\right)^2 - A^k_{,j}A^j_{,k}$. Using the method given in this example, apply it to the following expressions:

a. $\overbrace{\vec{\nabla}}^{\times}\underbrace{\vec{A}}_{\times}\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times} = \varepsilon^{ijk}\left(\varepsilon_{ilm}A^l A_{k,\cdot}{}^{m}\right)_{,j} = \delta^j_l\delta^k_m\left(A^l A_{k,\cdot}{}^{m}\right)_{,j} - \delta^j_m\delta^k_l\left(A^l A_{k,\cdot}{}^{m}\right)_{,j} = \left(A^j A_{k,\cdot}{}^{k}\right)_{,j} - \left(A^k A_{k,\cdot}{}^{j}\right)_{,j} = A^j_{,j}A_{k,\cdot}{}^{k} + A^j A_{k,\cdot j}{}^{k} - A^k_{,j}A_{k,\cdot}{}^{j} - A^k A_{k,\cdot j}{}^{j}$. In vector notation, the resulted expression can be written as $\overbrace{\vec{\nabla}}^{\times}\underbrace{\vec{A}}_{\times}\,\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times} = \left(\vec{\nabla}\cdot\vec{A}\right)^2 + \vec{A}\cdot\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\right) - A^k_{,j}A_{k,\cdot}{}^{j} - \vec{A}\cdot\left(\nabla^2\vec{A}\right)$.

b. $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{\nabla}}^{\times}\,\underbrace{\vec{A}}_{\times}\overbrace{\vec{A}}^{\times} = \varepsilon^{ijk}\varepsilon_{ilm}\left(A^m A_k\right)_{,\cdot j}^{\;\;l} = \left(\delta_l^j\delta_m^k - \delta_m^j\delta_l^k\right)\left(A^m A_k\right)_{,\cdot j}^{\;\;l} = \left(A^k A_k\right)_{,\cdot j}^{\;\;j} -$

$\left(A^j A_k\right)_{,\cdot j}^{\;\;k} = \nabla^2 A^2 - 2\vec{A}\cdot\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\right) - A_{,\cdot}^{j\,k}A_{k,j} - \left(\vec{\nabla}\cdot\vec{A}\right)^2.$

Note that vector-type notation is not sufficient to express some of the terms in these expressions—another reason to use index/tensor notation for its generality.

15.17 EXAMPLE: COVARIANT DERIVATIVES OF METRIC TENSORS

Show that the derivative of metric tensors (i.e., g_{ij} and g^{ij}) is equal to zero.

Solution:

Since metric tensor is a constant in the Cartesian coordinate system, hence its derivative is null and will be the same for any coordinate system. To show this, we use Equation 10.14, or $\left(g_{ij}\right)_{,k} = \dfrac{\partial g_{ij}}{\partial x^k} - \Gamma_{ik}^n g_{nj} - \Gamma_{jk}^n g_{ni}$. Now using Equation 10.19 we can expand the last two terms on the R.H.S, or $\left(g_{ij}\right)_{,k} = \dfrac{\partial g_{ij}}{\partial x^k} - \dfrac{1}{2}\left(\dfrac{\partial g_{ij}}{\partial x^k} + \dfrac{\partial g_{kj}}{\partial x^i} - \dfrac{\partial g_{ki}}{\partial x^j}\right) - \dfrac{1}{2}\left(\dfrac{\partial g_{ij}}{\partial x^k} + \dfrac{\partial g_{ki}}{\partial x^j} - \dfrac{\partial g_{kj}}{\partial x^i}\right)$. The terms on the R.H.S cancel out and the expression is equal to zero, hence $\left(g_{ij}\right)_{,k} = 0$, and similarly $\left(g^{ij}\right)_{,k} = 0$.

CHAPTER 16

EXERCISES

1. Write out the following expression in full detail for $N=3$ dimensions using Einstein summation convention:

   a. $A = A^{ij}\vec{e}_i\vec{e}_j$

   b. $\vec{A}\cdot\vec{B} = A^i B^j g_{ij}$

   c. $\nabla^2\psi = \frac{1}{\mathcal{J}}\frac{\partial}{\partial x^i}\left(\mathcal{J}g^{ij}\frac{\partial\psi}{\partial x^j}\right)$

   d. $C(i) = h_i\frac{e_{ijk}}{\mathcal{J}}\frac{\partial\left(h_k A(k)\right)}{\partial x^j}$

   e. $A^{ij}_{k,n} = \frac{\partial A^{ij}_k}{\partial x^n} + \Gamma^i_{nm}A^{jm}_k + \Gamma^j_{nm}A^{im}_k - \Gamma^m_{nk}A^{ij}_m$

2. By rotating bipolar coordinates about X-axis (the axis where foci are located on) we obtain the bi-spherical coordinate system, $\left(x^1,x^2,x^3\right)\equiv(\xi,\eta,\varphi)$ given in terms of Cartesian $\left(y^1,y^2,y^3\right)\equiv(X,Y,Z)$, as
$$\begin{cases} X = \dfrac{a\sin\xi\cos\varphi}{\cosh\eta - \cos\xi} \\ Y = \dfrac{a\sin\xi\sin\varphi}{\cosh\eta - \cos\xi} \\ Z = \dfrac{a\sinh\eta}{\cosh\eta - \cos\xi} \end{cases}.$$
Find the basis vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ for the bi-spherical coordinate system in terms of the Cartesian unit vectors. Also find the scale factors, unit vectors, metric tensors (covariant and contravariant), Jacobian, volume element, and Christoffel symbols of the 1st kind.

3. By rotating bipolar coordinates about Y-axis (the axis perpendicular to the line connecting the foci) we obtain a toroidal coordinate system, $\left(x^1,x^2,x^3\right)\equiv(\xi,\eta,\varphi)$ given in terms of Cartesian $\left(y^1,y^2,y^3\right)\equiv(X,Y,Z)$, as $\begin{cases} X=\dfrac{a\sinh\eta\,\cos\varphi}{\cosh\eta-\cos\xi} \\ Y=\dfrac{a\sinh\eta\,\sin\varphi}{\cosh\eta-\cos\xi} \\ Z=\dfrac{a\sin\xi}{\cosh\eta-\cos\xi} \end{cases}$. Find the basis vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ for the toroidal coordinate system in terms of the Cartesian unit vectors. Also find the scale factors, unit vectors, metric tensors (covariant and contravariant), Jacobian, volume element, and Christoffel symbols of the 2nd kind.

4. Having displacement vector $\vec{r}=y^i\vec{E}_i$ defined in Cartesian coordinate system y^i, express $\vec{r}$ in terms of the contravariant and physical components in cylindrical, spherical, bi-polar, bi-spherical, and toroidal coordinate systems. Use the definitions of these systems as given in previous examples and exercises.

5. Show that for an orthogonal coordinate system the relation $g_{ii}=1/g^{ii}=h_i^2$ holds, i.e., the diagonal elements of covariant and contravariant metric tensors. Note, no sum on index i.

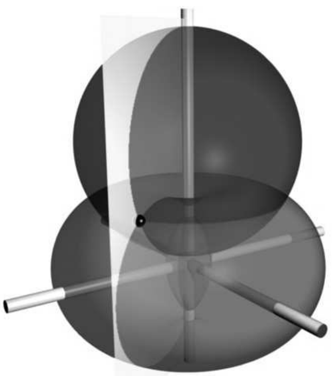


FIGURE 16.1 Bi-spherical coordinate system[1].

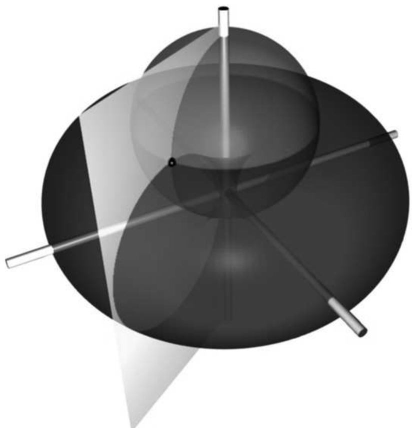


FIGURE 16.2 Toroidal coordinate system[2].

[1] <GNU Free Documentation License>
[2] <GNU Free Documentation License>

6. For an orthogonal coordinate system, show that the following relations holds for Christoffel symbols of the 2nd kind (all indices are set to be different):

$$\Gamma^k_{ij} = 0,\ \Gamma^j_{ii} = -\frac{h_i}{h_j^2}\frac{\partial h_i}{\partial x^j},\ \Gamma^i_{ij} = \frac{1}{h_i}\frac{\partial h_i}{\partial x^j},\ \Gamma^i_{ii} = \frac{1}{h_i}\frac{\partial h_i}{\partial x^i}.$$

7. Find the relations for Christoffel symbols of the 1st kind for cylindrical and spherical coordinate systems. Use results from the example given in Section 10.4.1.

8. Show that covariant differentiation obeys the same rule as does ordinary differentiation, when operating on products, e.g., $\left(A^iB^j\right)_{,k} = A^i_{,k}B^j + A^iB^j_{,k}$.

9. Write down the expression for the covariant derivative of a tensor of rank five, A^{ij}_{km}, using hints from Section 10.14.

10. Show that metric tensors behave like a constant under a covariant differentiation operation, i.e., $\left(A^ig_{ij}\right)_{,k} = A^i_{,k}g_{ij}$ and $\left(A_ig^{ij}\right)_{,k} = A_{i,k}g^{ij}$.

11. Show that the recursive relation for the curl of a vector $\vec{A}$ reads $\vec{A}(n) = -\nabla^2\vec{A}(n-2)$, for $n > 2$. where n is the number of curl operations and $\vec{A}(n) = \underbrace{\vec{\nabla}\times\vec{\nabla}\times\cdots\times\vec{\nabla}}_{n\ times}\times\vec{A}$ (note, $\vec{A}(0) = \vec{A}$).

12. Using the recursive relation for the curl of a vector given in Exercise 11, show that the recursive relation for the biharmonic operator reads $\vec{A}(n) = \nabla^4\vec{A}(n-4)$, for $n > 4$.

13. Using the results of Exercises 11 and 12, show that the recursive relations for higher order operator ∇^{2n} reads $\vec{A}(2n+1) = (-1)^n\nabla^{2n}\vec{A}(1)$ and $\vec{A}(2n) = -\nabla^{2n-2}\vec{A}(2)$.

14. For the 4D spherical coordinate system from Example 15.15, calculate the Christoffel symbols:

$$\Gamma^r = \begin{bmatrix} \Gamma^r_{rr} & \Gamma^r_{r\psi} & \Gamma^r_{r\theta} & \Gamma^r_{r\varphi} \\ \cdots & \Gamma^r_{\psi\psi} & \Gamma^r_{\psi\theta} & \Gamma^r_{\psi\varphi} \\ \cdots & \cdots & \Gamma^r_{\theta\theta} & \Gamma^r_{\theta\varphi} \\ \cdots & \cdots & \cdots & \Gamma^r_{\varphi\varphi} \end{bmatrix},\ \Gamma^\psi = \begin{bmatrix} \Gamma^\psi_{rr} & \Gamma^\psi_{r\psi} & \Gamma^\psi_{r\theta} & \Gamma^\psi_{r\varphi} \\ \cdots & \Gamma^\psi_{\psi\psi} & \Gamma^\psi_{\psi\theta} & \Gamma^\psi_{\psi\varphi} \\ \cdots & \cdots & \Gamma^\psi_{\theta\theta} & \Gamma^\psi_{\theta\varphi} \\ \cdots & \cdots & \cdots & \Gamma^\psi_{\varphi\varphi} \end{bmatrix},$$

$$\Gamma^{\theta}=\begin{bmatrix}\Gamma^{\theta}_{rr} & \Gamma^{\theta}_{r\psi} & \Gamma^{\theta}_{r\theta} & \Gamma^{\theta}_{r\varphi}\\ \cdots & \Gamma^{\theta}_{\psi\psi} & \Gamma^{\theta}_{\psi\theta} & \Gamma^{\theta}_{\psi\varphi}\\ \cdots & \cdots & \Gamma^{\theta}_{\theta\theta} & \Gamma^{\theta}_{\theta\varphi}\\ \cdots & \cdots & \cdots & \Gamma^{\theta}_{\varphi\varphi}\end{bmatrix}\quad \Gamma^{\varphi}=\begin{bmatrix}\Gamma^{\varphi}_{rr} & \Gamma^{\varphi}_{r\psi} & \Gamma^{\varphi}_{r\theta} & \Gamma^{\varphi}_{r\varphi}\\ \cdots & \Gamma^{\varphi}_{\psi\psi} & \Gamma^{\varphi}_{\psi\theta} & \Gamma^{\varphi}_{\psi\varphi}\\ \cdots & \cdots & \Gamma^{\varphi}_{\theta\theta} & \Gamma^{\varphi}_{\theta\varphi}\\ \cdots & \cdots & \cdots & \Gamma^{\varphi}_{\varphi\varphi}\end{bmatrix}.$$

15. Show that the following expressions simplify as given (see Example 15.16):

a. $\underbrace{\vec{\nabla}}_{\times}\underbrace{\vec{A}}_{\times}\overbrace{\vec{\nabla}}^{\times}\overbrace{\vec{A}}^{\times} = A_{k,j}A^{k\,j}_{,\,\cdot} + \vec{A}\cdot\left(\nabla^{2}\vec{A}\right) - A_{k,j}A^{j\,k}_{,\,\cdot} - \vec{A}\cdot\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\right)$

b. $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times}\,\overbrace{\vec{\nabla}}^{\times}\underbrace{\vec{\nabla}}_{\times}\Phi = \left(\vec{\nabla}\cdot\vec{A}\right)\left(\nabla^{2}\Phi\right) - A^{k}_{,\,j}\left(\nabla_{k}\Phi\right)^{\,j}_{,\cdot}$

c. $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{\nabla}}^{\times}\overbrace{\vec{A}}^{\times}\,\underbrace{\vec{\nabla}}_{\times}\Phi = \nabla^{2}\left(\vec{A}\cdot(\nabla\Phi)\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\cdot(\nabla\Phi) - A^{\,j\,k}_{,\,\cdot}\left(\nabla_{k}\Phi\right)$

$$-\left(\vec{\nabla}\cdot\vec{A}\right)\left(\nabla^{2}\Phi\right) - \vec{A}\cdot\left(\vec{\nabla}\left(\nabla^{2}\Phi\right)\right)$$

16. Write the below expressions in index notation:

a. $\left(\vec{V}\vec{\nabla}\right)\cdot\vec{A} + \left(\vec{\nabla}\cdot\vec{V}\right)\vec{A}$

b. $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)$

c. $\vec{A}\cdot\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\right)$

17. Show that the following relations holds:

a. $\varepsilon^{ijk}\varepsilon_{lmn} = \begin{vmatrix}\delta^{i}_{l} & \delta^{i}_{m} & \delta^{i}_{n}\\ \delta^{j}_{l} & \delta^{j}_{m} & \delta^{j}_{n}\\ \delta^{k}_{l} & \delta^{k}_{m} & \delta^{k}_{n}\end{vmatrix}$

b. $\varepsilon^{ijk}\varepsilon_{imn} = \delta^{j}_{m}\delta^{k}_{n} - \delta^{j}_{n}\delta^{k}_{m}$

c. $\varepsilon^{ijk}\varepsilon_{ijn} = 2\delta^{k}_{n}$

d. $\varepsilon^{ijk}\varepsilon_{ijk} = 6$

18. Using Equation 9.13, write the determinant for a 5×5 matrix.

19. Use Equation 13.19 to find physical components of the convective time derivative of a velocity vector, and express the results for (3D) (hint: see Example 15.14):

a. Cartesian coordinates

b. Cylindrical coordinates

c. Spherical coordinates

20. Show that the non-zero Christoffel symbols of the 1st kind for cylindrical and spherical coordinates read:

a. Cylindrical coordinates: $\Gamma_{r\theta,\theta}=r$ and $\Gamma_{\theta\theta,r}=-r$.

b. Spherical coordinates: $\Gamma_{\varphi\varphi,r}=-r$, $\Gamma_{\theta\theta,r}=-r\sin^2\varphi$, $\Gamma_{\theta\theta,\varphi}=-\dfrac{r^2}{2}\sin2\varphi$,

$\Gamma_{r\varphi,\varphi}=r$ and $\Gamma_{r\theta,\theta}=r\sin^2\varphi$, $\Gamma_{\varphi\theta,\theta}=\dfrac{r^2}{2}\sin2\varphi$

21. Using cross-product expression for two vectors, show that ($\mathcal{J}$ is the Jacobian):

a. $\mathcal{J}\mathcal{E}_{ijk}=\dfrac{1}{\mathcal{J}}\mathcal{E}^{mnl}g_{mi}g_{nj}g_{lk}$

b. $\mathcal{J}^2=\dfrac{1}{6}\mathcal{E}^{ijk}\mathcal{E}^{mnl}g_{mi}g_{nj}g_{kl}$

22. Calculate the single equivalent rotation angle and rotation axis for transforming Cartesian system (y_1,y_2,y_3) to (y_1',y_2',y_3') such that y_1' coincides with y_1, y_2' coincides with y_3, and y_3' coincides with y_1.

23. Find the rotation matrix for transforming (y_1,y_2,y_3):

a. Rotate about $(0,0,1)$ for $45°$

b. Then, rotate about $(1,0,0)$ for $60°$

c. Then, rotate about $(0,1,0)$ for $75°$

d. Find the final rotation matrix and equivalent rotation angle

e. Find equivalent axis of rotation

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